
Relativity implications of the quantum phase

Stephen G. Low

“So if one asks what is the main feature of quantum mechanics, I feel inclined now to say that it is not noncommutative algebra. It is the existence of probability amplitudes which underlie all atomic processes. Now a probability amplitude is related to experiment but only partially. The square of the modulus is something that we can observe. That is the probability which the experimental people get. But besides that there is a phase, a number of modulus unity which can modify without affecting the square of the modulus. And this phase is all important because it is the source of all interference phenomena but its physical significance is obscure.”

Dirac, 1972

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Preamble: Special relativistic quantum mechanics

Wigner has shown that special relativistic quantum mechanics can be understood in terms of the projective representations of the inhomogeneous Lorentz group*

Physical states Ψ are rays in a Hilbert space \mathbf{H} . Rays are equivalence classes of states

$$|\psi\rangle \simeq |\tilde{\psi}\rangle \in \Psi \quad \text{if} \quad |\tilde{\psi}\rangle = e^{i\omega} |\psi\rangle, \quad |\psi\rangle, |\tilde{\psi}\rangle \in \mathbf{H},$$

$$\|\Psi\|^2 = \langle \psi | \psi \rangle = \langle \tilde{\psi} | \tilde{\psi} \rangle$$

Representations of groups acting on the physical states are also defined up to a phase. These are projective representations.

The inhomogeneous Lorentz group is

$$\mathcal{IL}(1, n) \simeq \mathcal{L}(1, n) \otimes_s \mathcal{A}(n+1), \quad n = 3 \text{ physical case}$$

$\mathcal{L}(1, n)$ is the Lorentz group and $\mathcal{A}(n)$ is the abelian translation group $\mathcal{A}(n) \simeq (\mathbb{R}^n, +)$. It is a matrix group. $\Gamma \in \mathcal{IL}(1, n)$

$$\Gamma(\Lambda, a) = \begin{pmatrix} \Lambda & a \\ 0 & 1 \end{pmatrix}, \quad \Lambda^t \eta \Lambda = \eta, \quad a \in \mathbb{R}^{n+1}, \quad \eta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

*See for eg. Weinberg Chapter 2 of QFT volume I

Projective representations

The projective representations are calculated using a cornerstone theorem:

Theorem (Wigner, Bargmann, Mackey, Weinberg): A projective representation of a Lie group is equivalent to a unitary representation of the central extension of the group

Mackey has shown us how to compute the unitary representations of a general class of semidirect product groups

The central extension has an algebraic aspect and a topological aspect:

Algebraic: Construct the Lie algebra that results from adding the maximal number of central generators (that commute with all other generators) to the original Lie algebra without breaking the Jacobi identities

$$[X_\alpha, X_\beta] = c_{\alpha,\beta}^K X_K + M_{\alpha,\beta}, \quad [X_K, M_{\alpha,\beta}] = 0$$

Topological: Take the cover of the resulting group

It turns out that for the inhomogeneous Lorentz group, that you cannot add any algebraic central elements and so the central extension, that we call the Poincaré group, is just the cover

$$\overline{\mathcal{L}}(1, n) \simeq \mathcal{L}(1, n) \otimes_s \mathcal{A}(n + 1)$$

and $\overline{\mathcal{L}}(1, 3) \simeq \mathcal{SL}(2, \mathbb{C})$.

An aside: More on central extensions

Algebraic extensions occur in other cases. For example, in the nonrelativistic limit, the Lorentz group contracts to the inhomogeneous special orthogonal group

$$\mathcal{L}(1, n) \xrightarrow{c \rightarrow \infty} \mathcal{E}(n) \equiv \mathcal{ISO}(n) \simeq \mathcal{SO}(n) \otimes_s \mathcal{A}(n)$$

where $\Lambda^\circ \in \mathcal{ISO}(n)$

$$\Lambda^\circ = \begin{pmatrix} R & 0 \\ v & 1 \end{pmatrix}, \quad R \in \mathcal{SO}(n), \quad v \in \mathbb{R}^n$$

And the inhomogeneous Lorentz group becomes

$$\mathcal{IL}(1, n) \xrightarrow{c \rightarrow \infty} \mathcal{IE}(n) = \mathcal{ISO}(n) \otimes_s \mathcal{A}(n+1)$$

This group admits a central generator M that we identify with mass, and the central extension is the Galilei group.

The central extension of the abelian group by itself is

$$[X_\alpha, X_\beta] = 0 + M_{\alpha,\beta}, \quad [X_\kappa, M_{\alpha,\beta}] = 0$$

Jacobi identities are identically satisfied; the extension is $\frac{n(n-1)}{2}$ dimensional

$$[[X_\alpha, X_\beta], M_{\kappa,\gamma}] = 0, \quad [[X_\alpha, M_{\kappa,\gamma}], X_\beta] = 0 \quad \text{etc}$$

The Casimir invariants and wave equations

The Casimir invariants are what are invariant under the action of the group and the algebra

$$[X_\alpha, C_a] = 0$$

For $n = 3$, the Poincaré has two Casimirs

$$C_2 = P^2 = \eta^{a,b} P_a P_b, \quad C_4 = \eta^{a,b} W_a W_b$$

where

$$W_a = \epsilon_a^{b,c,d} L_{b,c} P_d,$$

and $\{L_{a,b}, P_a\}$ are generators of the algebra of the Poincaré group

Hermitian representation (that are observables) of the algebra correspond to the unitary representation of the central extension of the group

The representations of the Casimir's are also Hermitian operators and their Casimir's are mass and spin

$$\hat{C}_2 |\psi\rangle = \hat{P}^2 |\psi\rangle = m^2 |\psi\rangle, \quad \hat{C}_4 |\psi\rangle = m^2 s(s+1) |\psi\rangle$$

Mackey representations lead directly to the Klein-Gordon, Dirac, Maxwell, wave equations

Where is the Weyl-Heisenberg group and algebra?

This truly one of the most beautiful results in physics. From a basic matrix group and the observation that physical states are rays in a Hilbert case, out tumbles special relativistic quantum mechanics:

the Hilbert space of inertial states over the mass shells

the unitary representation of the group that transforms between inertial states

mass and spin as Casimir eigenvalues labeling the unitary irreducible representations

and the wave equations: Klein-Gordon, Dirac, Maxwell,

Except for one thing..... where is the Weyl-Heisenberg group?

What makes quantum mechanics, quantum mechanics?

The Heisenberg commutation relations are the Hermitian representation of the Lie algebra of the Weyl-Heisenberg group

$$[\hat{P}_i, \hat{Q}_j] = i \hbar \delta_{i,j} \hat{I}, \quad [\hat{E}, \hat{T}] = -i \hbar \hat{I}$$

and

$$\langle q, t | \hat{P}_i | \psi \rangle = i \hbar \frac{\partial}{\partial q^i} \psi(q, t), \quad \langle q, t | \hat{E} | \psi \rangle = -i \hbar \frac{\partial}{\partial t} \psi(q, t)$$

This has to be put in 'by hand' in the above derivation

The Weyl-Heisenberg group

The Weyl-Heisenberg group

$$\mathcal{H}(n) \simeq \mathcal{A}(n) \otimes_s \mathcal{A}(n+1)$$

is a real matrix group, $p, q \in \mathbb{R}^n$, $\iota \in \mathbb{R}$

$$Y(p, q, \iota) = \begin{pmatrix} 1_n & 0 & 0 & q \\ 0 & 1_n & 0 & p \\ p^t & -q^t & 1 & 2\iota \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{aligned} Y(0, q, \iota) &\in \mathcal{A}(n+1) \\ Y(p, 0, 0) &\in \mathcal{A}(n) \end{aligned}$$

Differentiate to get the matrix algebra

$$Z = q^i P_i + p^i Q_i + \iota I = \begin{pmatrix} 0 & 0 & 0 & q \\ 0 & 0 & 0 & p \\ p^t & -q^t & 0 & 2\iota \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad P = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

This satisfies the Weyl-Heisenberg algebra

$$[P_i, Q_j] = \delta_{i,j} I$$

The Weyl-Heisenberg group central extension

The algebra may be written with $\{X_\alpha\} = \{P_i, Q_i\}$ $\alpha, \beta = 1, \dots, 2n$

$$[X_\alpha, X_\beta] = \zeta_{\alpha,\beta} I, \quad [\zeta_{\alpha,\beta}] = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}$$

This is a 1-parameter central extension of the algebra of $\mathcal{A}(2n)$ that is the group of translations on phase space.

The full $\frac{n(n-1)}{2}$ dimensional central extension of $\mathcal{A}(2n)$ is

$$[X_\alpha, X_\beta] = M_{\alpha,\beta}, \quad [X_\kappa, M_{\alpha,\beta}] = 0$$

with the 1-parameter case

$$M_{\alpha,\beta} = \zeta_{\alpha,\beta} I.$$

Therefore, from the cornerstone theorem:

The unitary representations of the Weyl-Heisenberg group are a particular projective representation of the abelian translation group $\mathcal{A}(2n)$ on phase space

The projective representation “is quantization”.

Unitary representations of the Weyl-Heisenberg group

The unitary representations result can be calculated from the Mackey theorems

The Hilbert space is $L^2(\mathbb{R}^n, \mathbb{C})$ and the Hermitian representation of the algebra

$$\langle q | \hat{P}_i | \psi_t \rangle = i \hbar \frac{\partial}{\partial q^i} \psi_t(q), \quad \langle q | \hat{Q}_i | \psi_t \rangle = q_i \psi_t(q)$$

The 1 parameter central extension is not the most general projective representation.

What is constraining it?

Largest group consistent with Heisenberg commutation relations

A unitary operator \hat{U} representing a group element $U \in \mathcal{G}$ transforms states as

$$|\tilde{\psi}\rangle = \hat{U} |\psi\rangle$$

and a Hermitian operator transforms as

$$\hat{X}' = \hat{U} \hat{X} \hat{U}^{-1}$$

and in particular

$$\hat{P}'_i = \hat{U} \hat{P}_i \hat{U}^{-1}, \quad \hat{Q}'_j = \hat{U} \hat{Q}_j \hat{U}^{-1}, \quad \hat{I}' = \hat{U} \hat{I} \hat{U}^{-1} = \hat{I}$$

Now, we want the Heisenberg commutation relations to be preserved under the transformation so that the uncertainty principle is valid in all states related by U

$$[\hat{P}'_i, \hat{Q}'_j] = \hat{U} [\hat{P}_i, \hat{Q}_j] \hat{U}^{-1}$$

You can 'peel' off the representation and return to the finite real matrix Lie algebra

$$[P'_i, Q'_j] = U [P_i, Q_j] U^{-1}$$

This means that U is an element of the automorphism group $\mathcal{Aut}_{\mathcal{H}(n)}$ of the Weyl-Heisenberg group and algebra.

The Weyl-Heisenberg automorphism group

The automorphism group is

$$\mathcal{A}ut_{\mathcal{H}(n)} \simeq \mathcal{D} \otimes_s \overline{\mathcal{S}p}(2n) \otimes_s \mathcal{H}(n), \quad \mathcal{D} \simeq (\mathbb{R}^+, \times)$$

The inhomogeneous symplectic group is a symmetry of the phase space classical Hamilton's mechanics

$$\mathcal{I}Sp(2n) \simeq Sp(2n) \otimes_s \mathcal{A}(2n)$$

A calculation shows that its central extension (required for projective representations) is

$$\mathcal{I}\check{S}p(2n) \simeq \overline{\mathcal{S}p}(2n) \otimes_s \mathcal{H}(n)$$

As a result of the symplectic symmetry the most general representation of the abelian group is the unitary representations of the Weyl-Heisenberg group.

$$\mathcal{A}ut_{\mathcal{H}(n)} \simeq \mathcal{D} \otimes_s \mathcal{I}\check{S}p(2n)$$

The projective representations of $\mathcal{D} \otimes_s \mathcal{I}Sp(2n)$ are the unitary representations of $\mathcal{A}ut_{\mathcal{H}(n)}$

The Mackey nonabelian theorems show that the Hilbert space is $L^2(\mathbb{R}^n, \mathbf{H})$ with wave functions $\psi(q)$ or $\psi(p)$ not $\psi(q, p)$

Extended phase space

The exact same analysis applies on extended phase space where the Heisenberg commutation relations are

$$[\hat{P}_i, \hat{Q}_i] = i \hbar \delta_{i,j} \hat{I}, \quad [\hat{T}, \hat{E}] = i \hbar \hat{I}$$

The classical symmetry on extended phase space is

$$\mathcal{I}Sp(2n+2) \simeq Sp(2n+2) \otimes_s \mathcal{A}(2n+2)$$

These are the unitary representations of the central extension

$$\mathcal{A}ut_{\mathcal{H}(n+1)} \simeq \mathcal{D} \otimes_s \mathcal{I}\check{S}p(2n+2) \simeq \mathcal{D} \otimes_s \overline{Sp}(2n+2) \otimes_s \mathcal{H}(n+1)$$

$\mathcal{D} \otimes_s \mathcal{I}Sp(2n+2)$ is the largest group for which its projective representations define Heisenberg commutation relations that are preserved under all transformations of the representation acting on the Hilbert space

Both the Poincaré and Galilei groups are subgroups of $\mathcal{A}ut_{\mathcal{H}(n+1)}$ that transform between inertial states

But Poincaré and Galilei are not the most general relativity groups consistent with the Heisenberg commutation relations

The relativistic line elements and groups

The Minkowski metric is now degenerate line element on extended phase space $\mathbb{P} \simeq \mathbb{R}^{2n+2}$

$$d\tau^2 = dt^2 - \frac{1}{c^2} dq^2, \quad \begin{pmatrix} \eta & 0 \\ 0 & 0 \end{pmatrix}$$

just as Newtonian time is degenerate on spacetime and extended phase space

$$dt^2, \quad \begin{pmatrix} \eta^\circ & 0 \\ 0 & 0 \end{pmatrix} \quad \text{with } \eta^\circ = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

This leads us to conjecture the nondegenerate Born metric with signature $(2, 2n)$

$$ds^2 = dt^2 - \frac{1}{c^2} dq^2 + \frac{1}{b^2} \left(\frac{1}{c^2} de^2 - dp^2 \right), \quad \begin{pmatrix} \eta & 0 \\ 0 & \eta \end{pmatrix}$$

The homogeneous group that leaves the Born line elements invariant that is a subgroup of the Weyl-Heisenberg automorphism group is

$$\mathcal{U}(1, n) \simeq \mathcal{D} \otimes Sp(2n+2) \cap O(2, 2n)$$

This is the relativity group for any physical state, inertial or noninertial

The Lorentz inertial group is a subgroup $\mathcal{L}(1, n) \subset \mathcal{U}(1, n)$

Doesn't General Relativity address noninertial states?

A particle that is in an 'apparent' noninertial state due to the 'force' of gravity is actually in a locally inertial state on a curved space time

In a purely gravitating system, all particles follow geodesics that are locally inertial trajectories in this curved space-time

Neighboring clocks are related by the metric that now depends on the location in space-time

The connection translates between neighboring locally inertial frames

There are 'no forces' only geometry (Wheeler)

There are no noninertial states, only locally inertial states

But this does not address the noninertial state of a particle that is say an electron in a magnetic field

There is no known way to geometrize this force. (This has been searched for extensively).

How are the clocks of these noninertial states related?

This what reciprocal relativity addresses.

Reciprocal relativity: time dilation and null hypersurface

The Born metric on extended phase space $\mathbb{P} \simeq \mathbb{R}^{2n+2}$ is

$$d s^2 = d t^2 - \frac{1}{c^2} d q^2 + \frac{1}{b^2} \left(\frac{1}{c^2} d e^2 - d p^2 \right)$$

$$= d t^2 \left(1 - \frac{1}{c^2} v^2 + \frac{1}{b^2} \left(\frac{1}{c^2} r^2 - f^2 \right) \right)$$

velocity: $v^i = \frac{d q^i}{d t}$, power: $r = \frac{d e}{d t}$, force: $f^i = \frac{d p^i}{d t}$

The noninertial motion affects the clocks of an observer

$$d t = \frac{1}{\sqrt{1 - \frac{v^2}{c^2} - \frac{f^2}{b^2} + \frac{r^2}{b^2 c^2}}} d s$$

$\{c, b, \hbar\}$ are dimensional scales that give Planck scales $\{\lambda_t, \lambda_q, \lambda_e, \lambda_p\}$ by solving

$$\frac{\lambda_q}{\lambda_t} = c = \frac{\lambda_e}{\lambda_p}, \quad \lambda_q \lambda_p = \hbar = \lambda_t \lambda_e, \quad \frac{\lambda_p}{\lambda_t} = b = \frac{\lambda_e}{\lambda_q},$$

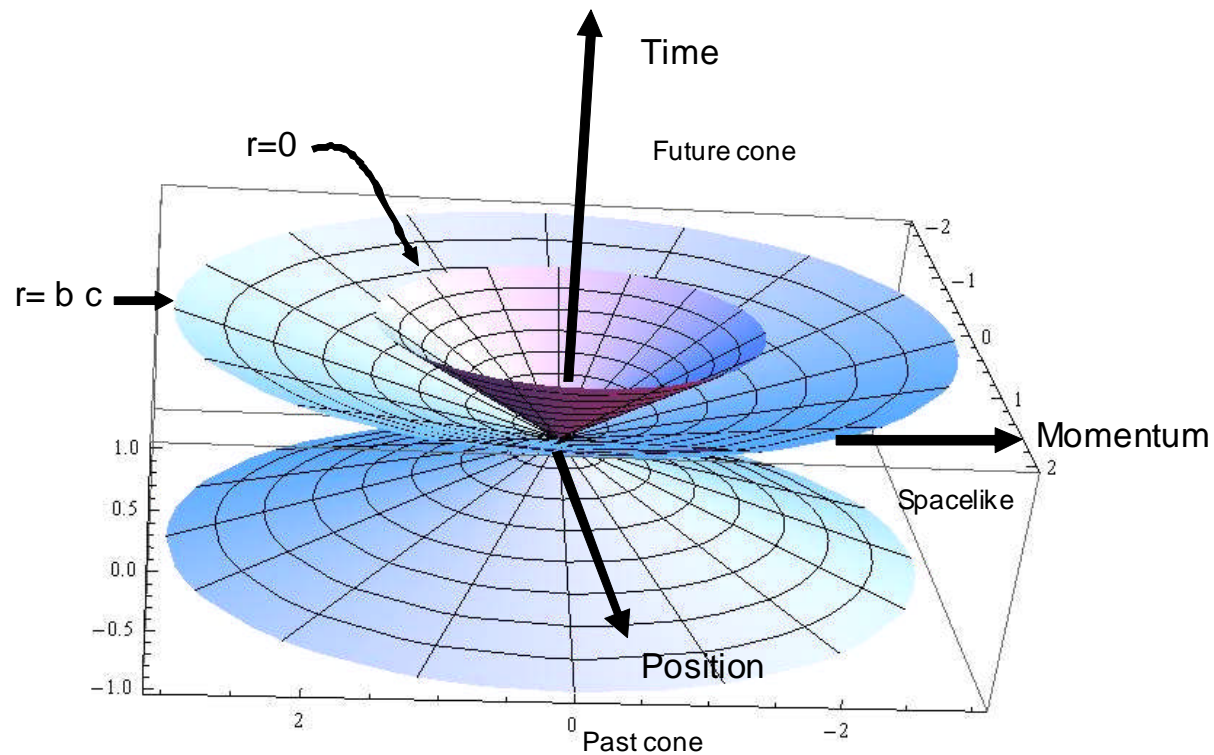
with $b = \frac{c^4}{G}$. In general $G = \alpha_G \frac{c^4}{b}$ with α_G free parameter of the theory

The null hypersurfaces that define the past and future cones are defined by

$$0 = 1 - \frac{v^2}{c^2} - \frac{f^2}{b^2} + \frac{r^2}{c^2 b^2}$$

Reciprocal relativity null hypersurface

$$\frac{v^2}{c^2} + \frac{f^2}{b^2} = 1 + \frac{r^2}{c^2 b^2}$$



Light is on the null cone

$v = dq/dt$ and $f = dp/dt$ are bounded

Cone flattens with increased r , increased noninertial

Relativity implications of the quantum phase

Quantum theory defined by the projective representations of the inhomogeneous unitary group

$$\mathcal{IU}(1, n) \simeq \mathcal{U}(1, n) \otimes_s \mathcal{A}(2n)$$

These are the unitary representations of the quaplectic group

$$\mathcal{Q}(1, n) = \mathcal{I}\check{\mathcal{U}}(1, n) = \overline{\mathcal{U}}(1, n) \otimes_s \mathcal{H}(n+1)$$

Requires the full power of the nonabelian Mackey theorems:

the Hilbert space of states is $L^2(\mathbb{R}^{n+1}, \mathbf{H})$

wave functions are functions of commuting subset of $\mathcal{H}(n+1)$,

not the full phase space

$\psi(t, q), \psi(e, p), \psi(t, p)$ or $\psi(e, q)$ “Not $\psi(t, q, e, p)$ ”

the mass shell is statistically determined by the wave function,

it is not a hypersurface

the unitary representations of the group transforms between noninertial states

the Hermitian representations of the Casimir invariants result in the wave equations

that are - ‘spinning relativistic oscillators’

The physical limits

This must yield the expected theory in the limit of small interactions, $b \rightarrow \infty$

The line elements and Inönü-Wigner group contractions are

	Born		Minkowski		Newton
Line element	$d s^2$	$\xrightarrow{b \rightarrow \infty}$	$d \tau^2$	$\xrightarrow{c \rightarrow \infty}$	$d t^2$
Inhomogeneous Group	$I\mathcal{U}(1, n)$	$\xrightarrow{b \rightarrow \infty}$	$I\mathcal{L}a(1, n)$	$\xrightarrow{c \rightarrow \infty}$	$I\mathcal{H}a(n)$
Inertial Subgroup	$I\mathcal{L}(1, n)$	$\xrightarrow{b \rightarrow \infty}$	$I\mathcal{L}(1, n)$	$\xrightarrow{c \rightarrow \infty}$	$I\mathcal{E}(n)$

where

$$\mathcal{L}a(1, n) \simeq \mathcal{L}(1, n) \otimes_s \mathcal{A}(m), \quad m = \frac{(n+1)(n+2)}{2}$$

$$\mathcal{H}a(n) \simeq \mathcal{S}O(n) \otimes_s \mathcal{H}(n)$$

Projective representations of $I\mathcal{L}a(1, n)$ are unitary representations of $\overline{\mathcal{L}a}(1, n) \otimes_s \mathcal{H}(n+1)$

The unitary irreducible representations contain the projective representations of $I\mathcal{L}(1, n)$

These are the representations that describe special relativistic quantum mechanics

Hamilton's mechanics results from the full contraction of small interaction force f/b and velocity v/c

Comment on quantum Hamilton group

We expect the quantum theory to be the projective representations of the Inhomogeneous Hamilton group

$$I\mathcal{H}a(n) \simeq \mathcal{H}a(n) \otimes_s \mathcal{A}(2n+2) \quad \text{where} \quad \mathcal{H}a(n) = \mathcal{S}O(n) \otimes_s \mathcal{H}(n)$$

These are the unitary representation of the central extension

$$I\check{\mathcal{H}}a(n) \simeq \check{\mathcal{H}}a(n) \otimes_s \mathcal{H}(n+1)$$

There are three central elements.

I from the extension of $\mathcal{A}(2n+2)$ to $\mathcal{H}(n+1)$

M that is mass, the Galilei group is the inertial subgroup of this group

A that has dimensions of 1/tension. What is this?

A Interacts through a non-inertial generalization to usual ‘nonrelativistic’ spin

It is a *reciprocal mass* that embodies energy $A b^2$ just as mass embodies energy $M c^2$

In the full reciprocal relativistic theory they *combine* into an *oscillation*

This is a definitive prediction of the theory - it is in the ‘nonrelativistic’ domain and is a ‘residue’ of the full theory- it should be possible to detect

A ‘Missed opportunity’ theorem of Hamilton’s equations

This is the $b, c \rightarrow \infty, \hbar \rightarrow 0$ limit.

Theorem: Let $\mathbb{P} \simeq \mathbb{R}^{2n+2}$ be extended phase space with a symplectic 2-form $\omega = -d e \wedge d t + d p_i \wedge d q^i$ and degenerate line element $\gamma^\circ = d t^2$

Then, a diffeomorphism φ on \mathbb{P} leaving invariant the symplectic 2-form and line element

$$\varphi^* \omega = \omega, \quad \varphi^* \gamma^\circ = \gamma^\circ$$

are Hamilton’s equations and these have a $\mathcal{HSp}(2n) \simeq Sp(2n) \otimes_s \mathcal{H}(n)$ symmetry.

Proof sketch: The Jacobian of the transformations φ must be elements of the group preserving the symplectic metric and line element.

Can show directly that the symplectic and affine invariance is

$$\mathcal{HSp}(2n) \simeq Sp(2n+2) \cap IGL(2n+1, \mathbb{R})$$

Hamilton’s equations follows directly from Jacobian $\left[\frac{\partial \varphi}{\partial z} \right] = \Gamma \in \mathcal{HSp}(2n)$

Dyson speaks of “missed opportunity” theorems in his 1972 Gibbs lecture. This fits his description.

PHY396T: Topics in Particle Physics

Applications of Lie groups in physics

Stephen Low, T/Th 1.5 hrs

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Topics include

Start with an introduction to Lie groups and algebras, including matrix groups, and examples relevant to physics

Includes automorphisms, semidirect products, central extensions, Casimir invariants, Inönü-Wigner contractions, etc

Representation theory for quantum physics: Projective representations, Mackey theorems for the unitary representations of semidirect product groups including:

Unitary representations of the Weyl-Heisenberg group, projective representations of the inhomogeneous Lorentz and inhomogeneous unitary groups

Equip you with the group theory and representation tools to understand, evaluate and contribute to new theories based on a group approach such as the theory outlined

About two thirds of the material will be established material and about one third research material

Assignments that work through the concepts. Goal is for the latter third of the course to have some tractable assignments with potential publishable content for papers

A final thought

“The miracle of the appropriateness of the language of mathematics for the formulation of the laws of physics is a wonderful gift which we neither understand nor deserve”.

E. Wigner, “The Unreasonable Effectiveness of Mathematics in the Natural Sciences”, *Comm. P. and App. Math.*, **13**, 1960

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Summary

Quantum phase requires projective representations

Projective representations are unitary representation of central extension

A one parameter central extension of the abelian translation group on phase space is the Weyl-Heisenberg group

The central extension of the inhomogeneous symplectic group is the automorphism group of the Weyl-Heisenberg group

Projective representations of abelian translations on phase space are the unitary representations of the Weyl-Heisenberg group

The unitary representations of the automorphism group gives the largest Hilbert space and unitary representations for which the Heisenberg commutation relations hold

Relativity, that defines time and simultaneity. requires a orthogonal line element.

The Minkowski line element is degenerate, the Born line element is nondegenerate

This results in a relativity theory of noninertial (and inertial) states

The quantum theory is the projective representation of the inhomogeneous unitary group

The ‘nonrelativistic’ classical limit is Hamilton’s equations that describe classical noninertial states

Quantum theory in the limit has central extensions of I , M , A and A is a testable prediction of the theory

Planck scales

The dimensional scale b appears in the Born line element

$$d s^2 = d t^2 - \frac{1}{c^2} d q^2 + \frac{1}{b^2} \left(\frac{1}{c^2} d e^2 - d p^2 \right)$$

The usual three dimensional scale constants are taken to be $\{c, G, \hbar\}$ in terms of which the Planck scales $\{\lambda_t, \lambda_q, \lambda_p, \lambda_e\}$ can be written

$$\lambda_t = \sqrt{\frac{G \hbar}{c^5}}, \lambda_q = \sqrt{\frac{\hbar G}{c^3}}, \lambda_p = \sqrt{\frac{\hbar c^3}{G}}, \lambda_e = \sqrt{\frac{\hbar c^5}{G}}$$

These Planck scales satisfy the relations

$$\frac{\lambda_q}{\lambda_t} = c = \frac{\lambda_e}{\lambda_p}, \quad \lambda_q \lambda_p = \hbar = \lambda_t \lambda_e, \quad \frac{\lambda_p}{\lambda_t} = \frac{c^4}{G} = \frac{\lambda_e}{\lambda_q}$$

from which the Planck scales may be derived.

We take $\{c, b, \hbar\}$ to be the three dimensional scale constants

$$\frac{\lambda_p}{\lambda_t} = b = \frac{\lambda_e}{\lambda_q}$$

from which the Planck scales can be derived and $G = \alpha_G \frac{c^4}{b}$

The Weyl-Heisenberg automorphism group

This is equivalent to the group automorphism, $\Upsilon, \Upsilon' \in \mathcal{H}(n)$,

$$\Upsilon' = U \Upsilon U^{-1}$$

Expanding

$$\begin{pmatrix} 1_n & 0 & 0 & q' \\ 0 & 1_n & 0 & p' \\ p'^t & -q'^t & 1 & 2\iota' \\ 0 & 0 & 0 & 1 \end{pmatrix} = U \begin{pmatrix} 1_n & 0 & 0 & q \\ 0 & 1_n & 0 & p \\ p^t & -q^t & 1 & 2\iota \\ 0 & 0 & 0 & 1 \end{pmatrix} U^{-1}$$

Then a simple matrix multiplication calculation results ‘schematically’

$$U = \begin{pmatrix} A & 0 & q \\ & 0 & p \\ p^t & -q^t & 1 & \iota \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad A \in Sp(2n)$$

(There is also conformal scaling and discrete symmetry)

Reciprocal relativity noninertial transformation equations

Observers that are in noninertial states measure different spacetime subspaces of extended phase space

Time, position, momentum and energy ‘mixed’ by the unitary transforms

Measurement of length, time, momentum, energy relative to noninertial state

The inertial rest frame is relative to the noninertial observer

$$\begin{aligned}
 d s^2 &= d \tau^2 + \frac{1}{b^2} c^2 d \mu^2 \\
 &= \eta_{a,b} d x^a d x^b + \frac{c^2}{b^2} \eta_{a,b} d p^a d p^b
 \end{aligned}$$

$$\begin{aligned}
 d \tilde{x}^a &= \Lambda_b^a d x^b - \frac{1}{b^2} M_b^a d p^b \\
 d \tilde{p}^a &= \Lambda_b^a d p^b + M_b^a d x^b, \quad \left(\begin{array}{cc} \Lambda & -\frac{1}{b^2} M \\ M & \Lambda \end{array} \right) \in \mathcal{U}(1, n),
 \end{aligned}$$

Inertial special case

$$\begin{aligned}
 d \tilde{x}^a &= \Lambda_b^a d x^b \\
 d \tilde{p}^a &= \Lambda_b^a d p^b \quad \left(\begin{array}{cc} \Lambda & 0 \\ 0 & \Lambda \end{array} \right) \in \mathcal{L}(1, n),
 \end{aligned}$$

The $\check{I}\check{H}a(n)$ Casimirs

$$C_1 = I, \quad C_2 = M, \quad C_3 = A,$$

$$C_4 = T T - I R,$$

$$C_5 = C^2 B_{i,j} B_{i,j},$$

where

$$C = C_2 C_3 + C_4 = -A M + T^2 - I R,$$

$$B_{i,j} = J_{i,j} + \frac{1}{C} D_{i,j}.$$

C is a Casimir invariant as any polynomial combinations of a Casimir is a Casimir and the $D_{i,j}$ are given by

$$D_{i,j} = A D_{i,j}^1 + M D_{i,j}^2 + R D_{i,j}^3 + I D_{i,j}^4 + T (D_{i,j}^5 + D_{i,j}^6)$$

where

$$D_{i,j}^1 = G_j P_i - G_i P_j, \quad D_{i,j}^2 = F_j Q_i - F_i Q_j,$$

$$D_{i,j}^3 = P_i Q_j - P_j Q_i, \quad D_{i,j}^4 = F_i G_j - F_j G_i,$$

$$D_{i,j}^5 = F_i P_j - F_j P_i, \quad D_{i,j}^6 = G_i Q_j - G_j Q_i.$$

In the limit $A \rightarrow \infty$, this reduces to the usual inertial Galilean Casimirs