

PHY396T

Applications of Lie groups and their
representations
to relativistic and quantum physics

Fall Semester T/Th 9:00-10:30

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Chapter 1 : Syllabus

Chapter 2 : Basic properties of Lie groups

2.1 Algebraic properties of a Group

2.1.1 Definition of a group

2.1.2 Some examples of groups

2.1.3 Homomorphisms

2.1.4 Normal Subgroups

2.1.5 Automorphism Group

2.1.6 Quotient Group

2.1.7 Direct and Semidirect Products

2.1.8 Center and central extensions

2.2 Finite and countable groups

2.3 Lie Groups

2.4 Connected Lie groups

2.5 General Linear groups

2.6 Matrix groups

2.7 Notes

Chapter 3: Matrix group examples

- 3.1 Introduction**
- 3.2 Connected General Linear Groups**
- 3.3 Special Linear group**
- 3.4 Abelian group**
- 3.5 Orthogonal group**
- 3.6 Symplectic group**
- 3.7 Unitary group**
- 3.8 Inhomogeneous general linear group**
- 3.9 Other inhomogeneous groups**
- 3.10 Notes**

Chapter 4: Weyl-Heisenberg group

- 4.1 Symplectic and affine invariance group**
- 4.2 Weyl-Heisenberg group**
- 4.3 Automorphism group of the Weyl-Heisenberg group**
- 4.4 Semidirect product groups with a Weyl-Heisenberg normal subgroup**
- 4.5 The polarized Weyl-Heisenberg group**
- 4.6 Degenerate orthogonal metric group**
- 4.7 Notes**

Chapter 5: Symmetry group of Hamilton's mechanics

- 5.1 Euclidean invariance of Newton's mechanics**
- 5.2 Symplectic symmetry of Hamilton's equations**
- 5.3 $HSp(2n)$ symmetry of Hamilton's equations**
- 5.4 Physical meaning of the theorem**
- 5.5 Lagrangian formulation**
- 5.6 Summary**
- 5.7 Notes**

Chapter 6 : Basic properties of Lie algebras

- 6.1 Algebraic definition**
- 6.2 Lie algebra of a matrix group**
- 6.3 General linear algebra**
- 6.4 Special Linear algebra**
- 6.5 Orthogonal algebra**
- 6.6 Symplectic algebra**
- 6.7 Unitary algebra**
- 6.8 Inhomogeneous algebras**
- 6.9 Inhomogeneous Euclidean algebra**
- 6.10 Weyl-Heisenberg algebra**
- 6.11 Semidirect sum algebras with a Weyl-Heisenberg ideal**
- 6.12 Degenerate orthogonal metric algebra**
- 6.13 Notes**

Chapter 7: Relativity groups

- 7.1 Recap of Hamilton equation symmetry**
- 7.2 Special relativity**
- 7.3 The fundamental dimensional constants**
- 7.4 Dimensionally scaled groups and transformation equations**
- 7.5 Noninertial reciprocal relativistic transformations**
- 7.6 Special relativity and nonrelativistic limits**
- 7.7 Proper time**
- 7.8 Null surfaces**
- 7.9 Rindler acceleration**
- 7.10 Notes**

Chapter 8: More properties of Lie algebras

8.1 Lie Algebra structure and classification

8.2 Baker Campbell Hausdorff

8.3 Enveloping Algebra and Casimir Invariants

8.4 Inonu-Wigner contraction

8.5 Notes

Chapter 12: Interlude chapter

9.0.1 Introduction to rule based symbolic computing

9.0.2 Lie algebra examples

9.0.3 Alternatives to hand coding LaTeX

Chapter 9: Central extension of a Lie Algebra

- 10.1 Central Extensions of a Lie Algebra**
- 10.2 Central extension of semisimple groups**
- 10.3 Central extension of the abelian algebra**
- 10.4 Central extension of the $\mathcal{GL}(n, \mathbb{R})$ and $I\mathcal{GL}(n, \mathbb{R})$ groups.**
- 10.5 Central extension of the inhomogeneous orthogonal algebra**
- 10.6 Central extension of the inhomogeneous symplectic algebra**
- 10.7 Central extension of the inhomogeneous unitary and O_a algebras**
- 10.8 Central extension of the inhomogeneous Euclidean algebra: the Galilei algebra**
- 10.9 Central extension of the Hamilton algebra**
- 10.10 Notes**

Chapter 10: Topology and the central extension of a Lie group

- 11.1 Topological groups**
- 11.2 Compact**
- 11.3 Connectedness**
- 11.4 Basic Homotopy: the fundamental group**
- 11.5 Definition of the central extension of a Lie group**
- 11.6 Universal cover of a Lie group**
- 11.7 Properties of the central extension of a Lie group**
- 11.8 Cohomology of the continuous central extension of a Lie group ##**
- 11.9 Extension of disconnected groups**
- 11.10 Notes**

Chapter 11: Matrix group topology examples

12.1 $\mathcal{A}(1)$ is the cover of $\mathcal{U}(1)$

12.2 $\mathcal{SL}(2, \mathbb{C})$ is the cover of $\mathcal{L}(1, 3)$

12.3 The Heisenberg group on the torus

12.4 Notes

Chapter 13: Introduction to group representations in quantum mechanics

13.1 Hilbert Space

13.2 Definition of a representation

13.3 Unitary representations

13.4 Hermitian representation of the algebra and physical observables

13.5 Projective representations in quantum mechanics

13.6 Notes

Chapter 14: Quantum Mechanics and the Weyl-Heisenberg group

14.1 Introduction

14.2 The Weyl-Heisenberg group

14.3 Largest group representation with Heisenberg commutation relations

14.4 Notes

Chapter 15: Unitary Irreducible representations and the Mackey theorems

15.1 Unitary Irreducible representations

15.2 Unitary representations of the abelian group and characters

15.3 Representations of Casimir invariants and the enveloping algebra

15.4 Mackey induction theorem

15.5 Mackey theorems: Abelian normal subgroup case

15.6 Notes

Chapter 16: Unitary representations of the Weyl-Heisenberg group and algebra

Chapter 17: Unitary representations of the Euclidean group

17.1 Euclidean group properties

17.2 The algebra of the Euclidean group

17.3 The little groups

17.4 Representation of the stabilizer group

17.5 Representation of the full group from Mackey induction

17.6 Representation of algebra and the Casimir wave equations

17.7 Notes

Chapter 18: Projective representations of the inhomogeneous Lorentz group and special relativistic quantum mechanics

18.1 Summary

18.2 The Lorentz group and its cover

18.3 Group properties of the inhomogeneous Lorentz group

18.4 Group properties of the Poincaré group

18.5 The algebra of the inhomogeneous Lorentz and Poincaré groups

18.6 Inhomogeneous Lorentz group unitary irreducible representations

18.7 Poincaré group unitary irreducible representations

18.8 Notes

Chapter 19: The Poincaré wave equations

19.1 Klein-Gordon

19.2 Weyl

19.3 Dirac

19.4 Maxwell

19.5 Notes

Chapter 19: Mid term

20.1 Example take-home midterm question

Chapter 21: Research topic: reciprocal relativity

21.1 Recap of special relativity

21.2 Recap of the maximal symmetry group for quantum mechanics

21.3 Reciprocal relativity

21.4 Reciprocal relativistic quantum mechanics

Chapter 22: Contraction sequences for $b \rightarrow \infty$, $c \rightarrow \infty$, $\hbar \rightarrow 0$ and physical meaning

- 22.1 $c \rightarrow \infty$ limit of special relativity: Galilean relativity**
- 22.2 Limits of reciprocal relativity**
- 22.3 $b \rightarrow \infty$ limit of reciprocal relativity: small interactions**
- 22.4 Proof $b \rightarrow \infty$ representations contain the SRQM representations**
- 22.5 $c, b \rightarrow \infty$ limit of reciprocal relativity: small interactions, velocities**
- 22.6 Tension: the third central element**
- 22.7 $c \rightarrow \infty$ dual limit of reciprocal relativity**
- 22.8 $c, b \rightarrow \infty, \hbar \rightarrow 0$ classical limit of reciprocal relativity: Hamilton's mechanics**

Chapter 23: Nonabelian Mackey theorem and the spinning quantum harmonic oscillator

23.1 Overview

23.2 Oscillator group and algebra

23.3 Automorphism group for the Oscillator group

23.4 Projective representations of the Oscillator group

23.5 The Unitary group

Chapter 24: Projective representations of the inhomogeneous unitary group and reciprocally relativistic quantum mechanics

24.1 The central extension: the quaplectic group

24.2 The nonabelian Mackey theorems applied to the quaplectic group

Chapter 25: Reciprocally relativistic wave equations

Chapter 26: Projective representations of the special relativistic $c \rightarrow \infty$ limit

Chapter 27: Projective representations of the nonrelativistic $b, c \rightarrow \infty$ limit

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Chapter 20: Work in progress stuff
