PHY396T

Applications of Lie groups and their representations to relativistic and quantum physics

Fall Semester T/Th 9:00-10:30

Stephen G. Low

Chapter 1 : Syllabus

Chapter 2 : Basic properties of Lie groups

2.1 Algebraic properties of a Group

2.1.1 Definition of a group

2.1.2 Some examples of groups

2.1.3 Homomorphisms

2.1.4 Normal Subgroups

2.1.5 Automorphism Group

2.1.6 Quotient Group

2.1.7 Direct and Semidirect Products

2.1.8 Center and central extensions

2.2 Finite and countable groups

2.3 Lie Groups

2.4 Connected Lie groups

2.5 General Linear groups

2.6 Matrix groups

2.7 Notes

Chapter 3: Matrix group examples

Chapter 4: Weyl-Heisenberg group

4.1 Symplectic and affine invariance group

4.2 Weyl-Heisenberg group

4.3 Automorphism group of the Weyl-Heisenberg group

4.4 Semidirect product groups with a Weyl-Heisenberg normal subgroup

4.5 The polarized Weyl-Heisenberg group

4.6 Degenerate orthogonal metric group

4.7 Notes

Chapter 5: Symmetry group of Hamilton's mechanics

Chapter 6 : Basic properties of Lie algebras

Chapter 7: Relativity groups

Chapter 8: More properties of Lie algebras

Chapter 12: Interlude chapter

9.0.1 Introduction to rule based symbolic computing

9.0.2 Lie algebra examples

9.0.3 Alternatives to hand coding LaTex

Chapter 9: Central extension of a Lie Algebra

Chapter 10: Topology and the central extension of a Lie group

11.1 Topological groups

11.2 Compact

11.3 Connectedness

11.4 Basic Homotopy: the fundamental group

11.5 Definition of the central extension of a Lie group

11.6 Universal cover of a Lie group

11.7 Properties of the central extension of a Lie group

11.8 Cohomology of the continuous central extension of a Lie group

11.9 Extension of disconnected groups

11.10 Notes

Chapter 11: Matrix group topology examples

Chapter 13: Introduction to group representations in quantum mechanics

13.1 Hilbert Space

13.2 Definition of a representation

13.3 Unitary representations

13.4 Hermitian representation of the algebra and physical observables

13.5 Projective representations in quantum mechanics

13.6 Notes

Chapter 14: Quantum Mechanics and the Weyl-Heisenberg group

Chapter 15: Unitary Irreducible representations and the Mackey theorems

15.1 Unitary Irreducible representations

15.2 Unitary representations of the abelian group and characters

15.3 Representations of Casimir invariants and the enveloping algebra

15.4 Mackey induction theorem

15.5 Mackey theorems: Abelian normal subgroup case

15.6 Notes

Chapter 16: Unitary representations of the Weyl-Heisenberg group and algebra

Chapter 17: Unitary representations of the Euclidean group

Chapter 18: Projective representations of the inhomogeneous Lorentz group and special relativistic quantum mechanics

18.1 Summary

18.2 The Lorentz group and its cover

18.3 Group properties of the inhomogeneous Lorentz group

18.4 Group properties of the Poincaré group

18.5 The algebra of the inhomogeneous Lorentz and Poincaré groups

18.6 Inhomogeneous Lorentz group unitary irreducible representations

18.7 Poincaré group unitary irreducible representations

18.8 Notes

Chapter 19: The Poincaré wave equations

20.1   Example take-home midterm question

Chapter 21: Research topic: reciprocal relativity

21.1 Recap of special relativity

21.2 Recap of the maximal symmetry group for quantum mechanics

21.3 Reciprocal relativity

21.4 Reciprocal relativistic quantum mechanics

Chapter 22: Contraction sequences for $b \to \infty$, $c \to \infty$, $\hbar \to 0$ and physical meaning

Chapter 23: Nonabelian Mackey theorem and the spinning quantum harmonic oscillator

23.1 Overview

23.2 Oscillator group and algebra

23.3 Automorphism group for the Oscillator group

23.4 Projective representations of the Oscillator group

23.5 The Unitary group

Chapter 24: Projective representations of the inhomogeneous unitary group and reciprocally relativistic quantum mechanics

24.1 The central extension: the quaplectic group

24.2 The nonabelian Mackey theorems applied to the quaplectic group

Chapter 25: Reciprocally relativistic wave equations

Chapter 26: Projective representations of the special relativistic $c \to \infty$ limit

Chapter 27: Projective representations of the nonrelativistic $b, c \to \infty$ limit

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Chapter 20: Work in progress stuff