

# Noninertial Symmetry of Hamilton's Mechanics

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Dedicated to Professor DeWitt-Morette

We present a new derivation of Hamilton's equations that shows that they have a symmetry group  $Sp(2n) \otimes \mathcal{H}(n)$ .  $Sp(2n)$  is the real noncompact symplectic group and  $\mathcal{H}(n)$  is mathematically a Weyl-Heisenberg group that is parameterized by velocity, force and power where power is the central element of the group.

## 1 Symmetry group theorem of Hamilton's equations

Let  $\mathbb{P} = \mathbb{R}^{2n+2}$  be an extended phase space with coordinates  $\{z^a\} = \{y^\alpha, e, t\}$  where  $a, b = 1, \dots, 2n+2$  and  $\alpha, \beta = 1, \dots, 2n$ . The  $2n$   $y$  coordinates may also be written  $\{y^\alpha\} = \{p^i, q^i\}$  with  $i, j = 1, \dots, n$ . In these coordinates, there is a symplectic metric that may be written in the forms

$$\omega = \zeta_{a,b} dz^a dz^b = \zeta_{\alpha,\beta} dy^\alpha dy^\beta - de \wedge dt = \delta_{i,j} dp^i \wedge dq^j - de \wedge dt. \quad (1)$$

The  $2n+2$  dimensional square matrix of components  $\zeta = [\zeta_{a,b}]$  is given by

$$\zeta = \begin{pmatrix} \zeta^\circ & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \zeta^\circ = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}, \quad (2)$$

and  $1_n$  is the unit  $n$  dimensional square matrix. Assume also that there is a degenerate orthogonal line element

$$\gamma^\circ = dt^2 = \eta^\circ_{a,b} dz^a dz^b, \quad (3)$$

where the  $\eta^\circ_{a,b}$  are the components of the  $2n+2$  dimensional square matrix that is zero except for a 1 in the lower right hand corner,

$$\eta^\circ = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (4)$$

As  $\mathbb{P} = \mathbb{R}^{2n+2}$ , the coordinates and the form of the symplectic metric (2) and degenerate orthogonal line element (4) are defined globally.

## 1.1 Theorem

Let  $z \in \mathbb{P}$  be extended phase space as defined above with symplectic metric  $\omega$  and degenerate orthogonal line element  $\gamma^\circ = d t^2$ . Let  $\rho$  be a diffeomorphism  $\rho: \mathbb{P} \rightarrow \mathbb{P}: z \mapsto \tilde{z} = \rho(z)$  that leaves invariant the symplectic metric,  $\omega = \rho^* \omega$  and the degenerate orthogonal line element,  $\gamma^\circ = \rho^* \gamma^\circ$ . Then,

A) the group of transformations on the cotangent space leaving the symplectic metric and degenerate orthogonal line element invariant is

$$\mathcal{HSp}(2n) \simeq Sp(2n) \otimes_s \mathcal{H}(n). \quad (5)$$

where  $\mathcal{H}(n)$  is the Weyl-Heisenberg group and  $Sp(2n)$  is the real noncompact symplectic group. (The notation is not completely standard, other notations are  $Sp(2n, \mathbb{R})$  [1] or  $Sp(n, n)$  [2].)

B) locally the diffeomorphisms  $\rho$  must have Jacobian matrices that are an element of  $\mathcal{HSp}(2n)$ ,

$$\left[ \frac{\partial \rho^a(z)}{\partial z^b} \right] = \Gamma(z) \in \mathcal{HSp}(2n) \quad \forall z \in \mathbb{P}, \quad (6)$$

and consequently have a particular functional form that are Hamilton's equations [3].

To make this more explicit, in component notation, the metric and line element transform under the mapping  $\tilde{z}^a = \rho^a(z)$  as

$$\omega = \zeta_{a,b} d \tilde{z}^a d \tilde{z}^b = \zeta_{a,b} \frac{\partial \rho^a(z)}{\partial z^c} \frac{\partial \rho^b(z)}{\partial z^d} d z^c d z^d$$

$$\gamma^\circ = \eta^\circ_{a,b} d \tilde{z}^a d \tilde{z}^b = \eta^\circ_{a,b} \frac{\partial \rho^a(z)}{\partial z^c} \frac{\partial \rho^b(z)}{\partial z^d} d z^c d z^d$$

and so for the metric and line element to be invariant, the Jacobian matrices must satisfy

$$\zeta_{c,d} = \zeta_{a,b} \frac{\partial \rho^a(z)}{\partial z^c} \frac{\partial \rho^b(z)}{\partial z^d} \quad (7)$$

$$\eta^\circ_{c,d} = \eta^\circ_{a,b} \frac{\partial \rho^a(z)}{\partial z^c} \frac{\partial \rho^b(z)}{\partial z^d} \quad (8)$$

The proof that follows first shows that the matrix  $\Gamma(z)$  that is defined in (6) and that satisfies these equations is an element of  $\mathcal{HSp}(2n)$  and then that (6) is Hamilton's equations.

## 1.2 Proof of Part A: Symmetry group is $\mathcal{HSp}(2n)$

The symplectic metric on extended phase space is invariant under the symplectic group  $Sp(2n+2)$  and the degenerate orthogonal line  $d t^2$  element is invariant under the affine group

$$IGL(2n+1, \mathbb{R}) \simeq GL(2n+1, \mathbb{R}) \otimes_s \mathcal{A}(2n+1), \quad \mathcal{A}(m) \simeq (\mathbb{R}^m, +). \quad (9)$$

We show in this section that the group that leaves both the symplectic metric  $\omega$  and the degenerate orthogonal metric  $\gamma^\circ$  is

$$\mathcal{HSp}(2n) \simeq Sp(2n+2) \cap \mathcal{IGL}(2n+1, \mathbb{R}). \quad (10)$$

The symplectic metric  $\omega$  given in (2) and degenerate orthogonal line element  $\gamma^\circ$  given in (4) may be written in matrix notation as

$$\omega = dz^t \zeta dz, \quad dt^2 = dz^t \eta^\circ dz, \quad (11)$$

Using matrix notation, a transformation of the basis is  $d\tilde{z} = \Gamma dz$ ,  $\Gamma \in \mathcal{GL}(2n+2, \mathbb{R})$ . It leaves invariant the symplectic metric if

$$\Gamma^t \zeta \Gamma = \zeta, \quad (12)$$

and the degenerate orthogonal line element is invariant if

$$\Gamma^t \eta^\circ \Gamma = \eta^\circ. \quad (13)$$

Expand the  $2n+2$  square matrix  $\Gamma$  as

$$\Gamma = \begin{pmatrix} \Sigma & b & w \\ c & a & r \\ d & g & \epsilon \end{pmatrix}, \quad (14)$$

where  $\Sigma$  is a  $2n$  dimensional square matrix,  $b, w \in \mathbb{R}^{2n}$  are column vectors,  $c, d \in \mathbb{R}^{2n}$  are row vectors and  $a, r, g, \epsilon \in \mathbb{R}$ . Then expanding the expression for the invariance of the  $\eta^\circ$ , (13), [4]

$$\begin{aligned} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} &= \begin{pmatrix} \Sigma^t & c^t & d^t \\ b^t & a & g \\ w^t & r & \epsilon \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \Sigma & b & w \\ c & a & r \\ d & g & \epsilon \end{pmatrix} \\ &= \begin{pmatrix} d^t d & d^t g & d^t \epsilon \\ g d & g^2 & g \epsilon \\ d \epsilon & g \epsilon & \epsilon^2 \end{pmatrix}. \end{aligned} \quad (15)$$

This identity requires  $d = g = 0$  and  $\epsilon = \pm 1$ . Applying this to (14), and computing the determinant

$$\text{Det } \Gamma = \text{Det} \begin{pmatrix} \Sigma & b & w \\ c & a & r \\ 0 & 0 & \epsilon \end{pmatrix} = \epsilon \text{Det} \begin{pmatrix} \Sigma & b \\ c & a \end{pmatrix} \neq 0,$$

and so  $\begin{pmatrix} \Sigma & b \\ c & a \end{pmatrix} \in \mathcal{GL}(2n+1, \mathbb{R})$  with  $(w, r) \in \mathbb{R}^{2n+1}$ . Therefore, it is the matrix realization of the time inversion discrete group semidirect the affine group

$$\mathbb{Z}_2 \otimes_s \mathcal{IGL}(2n+1, \mathbb{R}). \quad (16)$$

The time inversion does not impact this discussion and we can take the parameter  $\epsilon$  of  $\mathbb{Z}_2$  to be unity,  $\epsilon = 1$ .

Next, the symplectic invariance condition (12) requires that

$$\begin{aligned}
 \begin{pmatrix} \zeta^\circ & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} &= \begin{pmatrix} \Sigma^t & c^t & 0 \\ b^t & a & 0 \\ w^t & r & 1 \end{pmatrix} \begin{pmatrix} \zeta^\circ & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \Sigma & b & w \\ c & a & r \\ 0 & 0 & 1 \end{pmatrix} \\
 &= \begin{pmatrix} \Sigma^t \zeta^\circ \Sigma & \Sigma^t \zeta^\circ b & -c^t + \Sigma^t \zeta^\circ w \\ b^t \zeta^\circ \Sigma & 0 & -a + b^t \zeta^\circ w \\ c + w^t \zeta^\circ \Sigma & a + w^t \zeta^\circ b & 0 \end{pmatrix}.
 \end{aligned} \tag{17}$$

This identity is satisfied with

$$b = 0, \quad a = 1, \quad c = -w^t \zeta^\circ \Sigma, \quad \Sigma^t \zeta^\circ \Sigma = \zeta^\circ. \tag{18}$$

$\Gamma$  now has the form

$$\Gamma(\Sigma, w, r) = \begin{pmatrix} \Sigma & 0 & w \\ -w^t \zeta^\circ \Sigma & 1 & r \\ 0 & 0 & 1 \end{pmatrix}. \tag{19}$$

where  $\Sigma \in \mathcal{Sp}(2n)$ ,  $w \in \mathbb{R}^{2n}$  and  $r \in \mathbb{R}$ .

The group multiplication of the matrix group given by (19) is determined by matrix multiplication to be

$$\Gamma(\Sigma'', w'', r'') = \Gamma(\Sigma', w', r') \Gamma(\Sigma, w, r), \tag{20}$$

where

$$\begin{aligned}
 \Sigma'' &= \Sigma' \Sigma, \\
 w'' &= w' + \Sigma' w, \\
 r'' &= r' + r - w'^t \zeta^\circ \Sigma' w.
 \end{aligned} \tag{21}$$

and the inverse is determined by the matrix inverse to be

$$\Gamma^{-1}(\Sigma, w, r) = \Gamma(\Sigma^{-1}, -\Sigma^{-1} w, -r). \tag{22}$$

The following groups are subgroups [5]

$$\begin{aligned}
 \Gamma(\Sigma, 0, 0) &\in \mathcal{Sp}(2n), \\
 \Gamma(1_{2n}, w, r) &= \Upsilon(w, r) \in \mathcal{H}(n) \simeq \mathcal{A}(n) \otimes_s \mathcal{A}(n+1).
 \end{aligned} \tag{23}$$

where  $\mathcal{A}(m)$  is the real abelian group under addition,  $\mathcal{A}(m) \simeq (\mathbb{R}^m, +)$ . It is then be shown that  $\Upsilon(w, r) \in \mathcal{H}(n)$  is a normal subgroup by computing the automorphisms

$$\begin{aligned}
 \Upsilon(w'', r'') &= \Gamma(\Sigma', w', r') \Upsilon(w, r) \Gamma^{-1}(\Sigma', w', r') \\
 &= \Upsilon(\Sigma' w, r + (\Sigma' w)^t \zeta^\circ w' - w'^t \zeta^\circ \Sigma' w).
 \end{aligned} \tag{24}$$

As the intersection of the subgroups in (23) is the identity and their product is the full group, it follows that the group is the semidirect product (5) as claimed.

That  $\mathcal{H}(n)$  is the Weyl-Heisenberg group may be determined by computing its algebra

$$W_a = \frac{\partial}{\partial w^a} \Upsilon(w, r) \Big|_{w=r=0}, \quad U = \frac{\partial}{\partial r} \Upsilon(w, r) \Big|_{w=r=0}. \tag{25}$$

A general element of the algebra is  $Z = w^\alpha W_\alpha + rU$ . The Lie algebra of a matrix group is the matrix commutators  $[A, B] = AB - BA$  that give

$$[W_\alpha, W_\beta] = 2\zeta^\circ_{\alpha,\beta} U, \quad [W_\alpha, U] = 0. \quad (26)$$

This is the Weyl-Heisenberg algebra where  $U$  is the central generator.

This completes the proof of Part A of the theorem that establishes that the group that has both symplectic and affine symmetry is  $\mathcal{HSp}(2n)$ .

### 1.3 Proof of Part B: Diffeomorphisms satisfy Hamilton's equations

The Jacobian matrix  $\left[\frac{\partial\rho(z)}{\partial z}\right]$  of the diffeomorphism  $\rho$  that leaves invariant the symplectic metric (3) and the degenerate orthogonal line element (1) must satisfy (12) and (13). Therefore, the Jacobian matrix is an element of the symmetry group,  $\left[\frac{\partial\rho(z)}{\partial z}\right] = \Gamma(z) \in \mathcal{HSp}(n)$ . Using the expanded notation with  $\{z\} = \{y, e, t\}$  with  $z \in \mathbb{R}^{2n+2}$ ,  $y \in \mathbb{P}^\circ \simeq \mathbb{R}^{2n}$  and  $e, t \in \mathbb{R}$ , the Jacobian matrix is

$$\begin{pmatrix} \frac{\partial\rho_y(y,e,t)}{\partial y} & \frac{\partial\rho_y(y,e,t)}{\partial e} & \frac{\partial\rho_y(y,e,t)}{\partial t} \\ \frac{\partial\rho_e(y,e,t)}{\partial y} & \frac{\partial\rho_e(y,e,t)}{\partial e} & \frac{\partial\rho_e(y,e,t)}{\partial t} \\ \frac{\partial\rho_t(y,e,t)}{\partial y} & \frac{\partial\rho_t(y,e,t)}{\partial e} & \frac{\partial\rho_t(y,e,t)}{\partial t} \end{pmatrix} = \begin{pmatrix} \Sigma(z) & 0 & w(z) \\ -w^t(z)\zeta^\circ & \Sigma(z) & 1 \\ 0 & 0 & 1 \end{pmatrix}. \quad (27)$$

where we have again suppressed indices using matrix notation.

This restricts the functional dependency of the diffeomorphisms as follows. First the time component,  $\frac{\partial\rho_t(y,e,t)}{\partial y} = \frac{\partial\rho_t(y,e,t)}{\partial e} = 0$  and  $\frac{\partial\rho_t(y,e,t)}{\partial t} = 1$  and so ignoring trivial integration constants,  $\rho_t(y, e, t) = t$ . Next for the energy component, note that  $\frac{\partial\rho_e(y,e,t)}{\partial e} = 1$  and therefore  $\rho_e$  may be written as  $\rho_e(y, e, t) = e + H(y, t)$  where  $H$  is some function. Finally,  $\frac{\partial\rho_y(y,e,t)}{\partial e} = 0$  and consequently  $\rho_y(y, e, t) = \varphi(y, t)$  where  $\varphi$  is some function.

Summarizing, the diffeomorphism  $\tilde{z} = \rho(z)$  can be expanded as

$$\begin{aligned} \tilde{y} &= \rho_y(y, e, t) = \varphi(y, t) = \phi_y(t), \\ \tilde{e} &= \rho_e(y, e, t) = e + H(y, t), \\ \tilde{t} &= \rho_t(y, e, t) = t. \end{aligned} \quad (28)$$

$H$  and  $\varphi$  are functions

$$\begin{aligned} H &: \mathbb{R}^{2n+1} \rightarrow \mathbb{R} : (y, t) \mapsto H(y, t), \\ \varphi &: \mathbb{R}^{2n+1} \rightarrow \mathbb{P}^\circ : (y, t) \mapsto \varphi(y, t). \end{aligned} \quad (29)$$

and  $\phi_y$  may be regarded as the curves

$$\phi_y : \mathbb{R} \rightarrow \mathbb{P}^\circ : t \mapsto \phi_y(t) \quad (30)$$

$H$  will turn out to be the Hamiltonian and  $\phi_y$  the curves that are the trajectories in phase space that are solutions to Hamilton's equations.

Substituting these back into (27), the Jacobian now has the form

$$\begin{pmatrix} \frac{\partial \varphi(y,t)}{\partial y} & 0 & \frac{\partial \varphi(y,t)}{\partial t} \\ \frac{\partial H(y,t)}{\partial y} & 1 & \frac{\partial H(y,t)}{\partial t} \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \Sigma(y,t) & 0 & w(y,t) \\ -w^t(y,t) \zeta^\circ & \Sigma(y,t) & 1 & r(y,t) \\ 0 & & 0 & 1 \end{pmatrix}. \quad (31)$$

Therefore we have

$$\frac{\partial \varphi(y,t)}{\partial y} = \Sigma(y,t), \quad \frac{\partial H(y,t)}{\partial y} = - \left[ \frac{\partial \varphi(y,t)}{\partial t} \right]^t \zeta^\circ \Sigma(y,t), \quad \frac{\partial H(y,t)}{\partial t} = r(y,t) \quad (32)$$

As  $\varphi(y,t)$  is a canonical transformation for some  $y^\circ$ ,  $y = \varphi(y^\circ, t)$  and for some  $t^\circ$ ,  $y^\circ = \phi_{y^\circ}(t^\circ)$  with  $\Sigma(y^\circ, t^\circ) = 1_{2n}$ . Then from the chain rule,

$$\frac{\partial \varphi(y,t)}{\partial t} = \frac{\partial \varphi(y,t)}{\partial y} \frac{\partial \varphi(y^\circ, t)}{\partial t} = \Sigma(y,t) \frac{\partial \varphi(y^\circ, t)}{\partial t} = \Sigma(y,t) \frac{d \phi_{y^\circ}(t)}{d t} \quad (33)$$

Consequently

$$\frac{\partial H(y,t)}{\partial y} = - \left[ \frac{d \phi_{y^\circ}(t)}{d t} \right]^t \Sigma^t(y,t) \zeta^\circ \Sigma(y,t) = - \left[ \frac{d \phi_{y^\circ}(t)}{d t} \right]^t \zeta^\circ \quad (34)$$

Re-arranging

$$\frac{d \phi_{y^\circ}(t)}{d t} = - \zeta^\circ \left[ \frac{\partial H(y,t)}{\partial y} \right]^t, \quad \frac{\partial H(y,t)}{\partial t} = r(y,t) \quad (35)$$

In component form this is

$$\frac{d \phi_{y^\circ}^\alpha(t)}{d t} = \zeta^{\circ\alpha,\beta} \frac{\partial H(y,t)}{\partial y^\beta}, \quad \frac{\partial H(y,t)}{\partial t} = r(y,t) \quad (36)$$

where  $[\zeta^{\circ\alpha,\beta}] = -\zeta^\circ$ . These are Hamilton's equations with the initial point  $y^\circ = \phi_{y^\circ}(t^\circ)$ .

The converse requires us to prove that if the diffeomorphisms satisfy Hamilton's equations (36), then the symplectic and line element are invariant.

$$\begin{aligned} \tilde{\omega} &= d \tilde{y}^t \zeta^\circ d \tilde{y} + d \tilde{t} \wedge d \tilde{e} \\ &= (d y + d \phi_{y^\circ}(t))^t \zeta^\circ (d y + d \phi_{y^\circ}(t)) + d t \wedge (d e + d H(y,t)) \\ &= d y^t \zeta^\circ d y + d t \wedge d e - \left[ \frac{d \phi_{y^\circ}(t)}{d t} \right]^t \zeta^\circ d y \wedge d t - \frac{\partial H(y,t)}{\partial y} d y \wedge d t \\ &= \omega - \left( \left[ \zeta^\circ \frac{d \phi_{y^\circ}(t)}{d t} \right]^t - \frac{\partial H(y,t)}{\partial y} \right) d y \wedge d t \\ &= \omega \end{aligned} \quad (37)$$

$\gamma^\circ = d t^2$  is invariant as  $t$  is an invariant parameter in Hamilton's equations. This completes the proof of the theorem.

A corollary of the theorem is that Hamilton's equations are valid in any extended canonical coordinates where the symplectic metric and degenerate line element have the form given in (1) and (3). Furthermore, transformations between these extended canonical coordinates must have a Jacobian that is an element of the  $\mathcal{HSp}(2n)$  group (6).

## 2 Physical meaning of the theorem

An element  $\Gamma \in \mathcal{HSp}(2n)$  can always be written as the product of a symplectic transformation and a Weyl-Heisenberg transformation

$$\Gamma(\Sigma, \Sigma y, r) = \Gamma(\Sigma, 0, 0) \Gamma(1_n, y, r). \quad (38)$$

We will consider the symplectic group first and show that this is the standard canonical transforms on phase space. Next, we consider the Weyl-Heisenberg transformations and show that they lead to familiar results.

### 2.1 Symplectic transformations

Consider first the symplectic transformations. In this case, the general transformations (28) reduce to

$$\tilde{y} = \rho_y(y, t) = \varphi(y, t), \quad \tilde{e} = \rho_e(e) = e, \quad \tilde{t} = \rho_t(t) = t, \quad (39)$$

with Jacobian satisfying

$$d \tilde{y} = \frac{\partial \varphi(y, t)}{\partial y} d y = \Sigma(y, t) d y. \quad (40)$$

The  $\varphi(y, t)$  are time dependent canonical transformations that appear in all the standard treatments of Hamilton's mechanics [6]. They may be regarded as the canonical transformations parameterized by time on the momentum, position phase space  $y \in \mathbb{P}^\circ \simeq \mathbb{R}^{2n}$

$$\varphi_t : \mathbb{P}^\circ \rightarrow \mathbb{P}^\circ : y \mapsto \tilde{y} = \varphi_t(y), \quad (41)$$

or as the curves  $\phi_y : \mathbb{R} \rightarrow \mathbb{P}^\circ$  that are given in (30). The solutions  $\phi_y$  to Hamilton's equations may be regarded as a time evolving canonical transformation.

The coordinates in which the symplectic metric have the canonical form (12) are canonical coordinates. In particular, Hamilton's equations are valid in any canonical coordinates  $\tilde{y} = \varrho(y)$  with

$$d \tilde{y} = \frac{\partial \varrho(y)}{\partial y} d y = \Sigma(y) d y. \quad (42)$$

Hamilton's equations in the tilde coordinates are

$$\frac{d \tilde{\phi}_{y^\circ}(t)}{d t} = -\zeta^\circ \left[ \frac{\partial \tilde{H}(\tilde{y}, t)}{\partial \tilde{y}} \right]^\dagger, \quad (43)$$

with

$$\tilde{H}(\tilde{y}, t) = \tilde{H}(\varrho(y), t) = H(y, t), \quad \tilde{\phi}_{y^\circ}(t) = \varrho(\phi_{y^\circ}(t)), \quad (44)$$

and therefore

$$\tilde{H} = H \circ \varrho^{-1} \text{ and } \tilde{\phi}_{y^\circ} = \varrho \circ \phi_{y^\circ}. \quad (45)$$

It then follows from the methods used to prove the general theorem that Hamilton's equations transform into the non-tilde coordinates for the transforms  $\varrho$  that are a special case of the more general  $\rho$  transforms of the theorem.

Note particularly that under a canonical transformation, that the Hamiltonian transforms as  $\tilde{H} = H \circ \varrho^{-1}$  given in (45) and not as an invariant function  $\tilde{H} = H$ . Canonical coordinates do not have the concept of states being inertial or noninertial and Hamilton's equations is valid in either provided that the Hamiltonian  $H(y, t)$  is chosen appropriately according to (45).

The phase space  $\mathbb{P}^\circ$  may be generalized to symplectic manifolds with Hamilton's equations expressed as the flows of Hamiltonian vector fields [7].

## 2.2 Weyl-Heisenberg transformations

Define  $y = (p, q)$ ,  $p, q \in \mathbb{R}^n$  and  $\phi = (\pi, \xi)$ , In components, this is  $\{y^a\} = \{p^i, q^i\}$ ,  $\{\phi^a(t)\} = \{\pi^i(t), \xi^i(t)\}$   $i, j = 1, \dots, n$ . As is usual,  $p$  is canonical momentum and  $q$  is canonical position. We will continue to use matrix notation with indices suppressed. Hamilton's equations then take on their most simple form,

$$\frac{d\xi(t)}{dt} = v = \frac{\partial H(p, q, t)}{\partial p}, \quad \frac{d\pi(t)}{dt} = f = -\frac{\partial H(p, q, t)}{\partial q}, \quad \frac{\partial H(p, q, t)}{\partial t} = r, \quad (46)$$

where  $v(p, q, t)$ ,  $f(p, q, t) \in \mathbb{R}^n$  are the velocity and force respectively and  $r(p, q, t) \in \mathbb{R}$  is the power. The velocity force and power are generally functions of  $(p, q, t)$  and this will be implicit in the following. The Weyl-Heisenberg subgroup may be written as

$$Y(f, v, r) = \Gamma(1_{2n}, f, v, r) = \begin{pmatrix} 1_n & 0 & 0 & f \\ 0 & 1_n & 1 & v \\ v & -f & 1 & r \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (47)$$

The coordinates  $z$  of the extended phase space  $\mathbb{P}$  may be similarly expanded as  $z = (p, q, e, t)$  and the Weyl-Heisenberg transformation  $d\tilde{z} = Y dz$  expands as

$$\begin{pmatrix} d\tilde{p} \\ d\tilde{q} \\ d\tilde{e} \\ d\tilde{t} \end{pmatrix} = \begin{pmatrix} 1_n & 0 & 0 & f \\ 0 & 1_n & 1 & v \\ v & -f & 1 & r \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} dp \\ dq \\ de \\ dt \end{pmatrix}. \quad (48)$$

Using Hamilton's equations (46), this results in

$$\begin{aligned} d\tilde{t} &= dt, \\ d\tilde{q} &= dq + v dt &&= dq + d\xi(t), \\ d\tilde{p} &= dp + f dt &&= dp + d\pi(t), \\ d\tilde{e} &= de + v \cdot dp - f \cdot dq + r dt &&= de + dH(p, q, t). \end{aligned} \quad (49)$$

These are the transformations that relate two states in extended phase space that have a relative rate of change of position, momentum and energy with respect to time. That is, they have a relative velocity  $v$ , force  $f$  and power  $r$ . These are general states in the extended phase space that may be inertial or noninertial. In the energy transformation,  $\int v \cdot dp$  is the incremental kinetic energy and  $-\int f \cdot dq$  is the work transforming from energy

state  $e$  to  $\tilde{e}$ . The term  $\int r dt$  is the explicit power for time dependent Hamiltonians. Solving Hamilton's equations enables these to be integrated to the form that is a special case of (28) with  $\Sigma = 1_{2n}$ ,

$$\begin{aligned}\tilde{t} &= \rho_t(t) = t, \\ \tilde{q} &= \rho_q(q, t) = q + \xi(t), \\ \tilde{p} &= \rho_p(p, t) = p + \pi(t), \\ \tilde{e} &= \rho_e(e, p, q, t) = e + H(p, q, t).\end{aligned}\tag{50}$$

Using the group multiplication (20-21) with  $\Sigma = 1_{2n}$ , or simply multiplying the matrices in (48) together shows that

$$\Upsilon(\tilde{f}, \tilde{v}, \tilde{r}) \Upsilon(f, v, r) = \Upsilon(f + \tilde{f}, v + \tilde{v}, r + \tilde{f}v - \tilde{v}f),\tag{51}$$

$$\Upsilon(f, v, r) \Upsilon(\tilde{f}, \tilde{v}, \tilde{r}) = \Upsilon(f + \tilde{f}, v + \tilde{v}, r - \tilde{f}v + \tilde{v}f).\tag{52}$$

These are not equal and consequently the operations do not commute. This can be made even more explicit by considering the case of a transformation in velocity followed by a transformation in force

$$\Upsilon(\tilde{f}, 0, 0) \Upsilon(0, v, 0) = \Upsilon(\tilde{f}, v, \tilde{f}v)\tag{53}$$

$$\Upsilon(0, v, 0) \Upsilon(\tilde{f}, 0, 0) = \Upsilon(\tilde{f}, v, -\tilde{f}v)\tag{54}$$

This is not unexpected. We do not expect an inertial transformation in velocity followed by a noninertial transformation in force to be the same as the noninertial force transformation followed by the inertial velocity transformation. This noncommutativity is why the nonabelian Weyl-Heisenberg group appears. The noncommutativity is also why noninertial states and frames are difficult to work with.

### 3 Discussion

Hamilton's mechanics is a reformulation of Newton's mechanics and is therefore invariant under Galilean relativity. The homogeneous relativity group is the Euclidean group  $\mathcal{E}(n) \simeq \mathcal{SO}(n) \otimes_s \mathcal{A}(n)$  parameterized by rotations and velocity. This is a subgroup of the group of transformations  $\mathcal{HSp}(2n)$ . The orthogonal group  $\mathcal{SO}(n) \subset \mathcal{Sp}(2n)$  where in this case the symplectic transformations on  $\mathbb{P}^o$  are just the rotations

$$\Sigma(R) = \begin{pmatrix} R & 0 \\ 0 & R \end{pmatrix}.\tag{55}$$

The space time translations are a subgroup of the Weyl-Heisenberg group,  $\mathcal{A}(n) \subset \mathcal{H}(n) \simeq \mathcal{A}(n) \otimes_s \mathcal{A}(n+1)$ . The resulting transformations are the inertial transformations on extended phase space

$$\begin{aligned}d\tilde{t} &= dt, \\ d\tilde{q} &= R dq + v dt, \\ d\tilde{p} &= R dp, \\ d\tilde{e} &= de + v \cdot dp.\end{aligned}\tag{56}$$

But why select this particular special case of the general  $\mathcal{H}Sp(2n)$  symmetry and give it the elevated status of a relativity group? Up to this point we have not made any comment on the particular functional form of the Hamiltonian  $H(p, q, t)$ . The theorem is silent on its form.

Physical considerations lead to Hamiltonians of many forms. For nonrelativistic electrodynamic, it is

$$H(p, q, t) = \frac{1}{2m} \left( p - \frac{\epsilon}{c} A(q, t) \right)^2 + \epsilon \phi(q, t) \quad (57)$$

where in this equation  $\phi(q, t)$  is the electric potential and  $\epsilon$  is the charge. The canonical momentum is related to the velocity through the expression

$$v(p, q, t) = \frac{p}{m} - \frac{\epsilon}{m c} A(q, t) \quad (58)$$

and so the relationship between velocity and momentum may be quite complex

For a broad class of problems in elementary classical mechanics, the Hamiltonian is given simply by

$$H(p, q, t) = K(p) + V(q) = \frac{p^2}{2m} + V(q). \quad (59)$$

Hamilton's equations result in  $v = \frac{p}{m}$  and  $\int v \cdot dp = \frac{p^2}{2m}$  is the kinetic energy  $K(p)$  and  $-\int f \cdot dq = V(q)$  is the potential energy. Energy is constant in time as  $\frac{\partial}{\partial t} H(p, q) = 0$ . This is but a most basic solution. An even more basic case is the inertial state where  $f = r = 0$  and therefore  $V(q) = 0$ . This state has the property that, from (49),

$$\tilde{H}(\tilde{p}) = H(p) + v \cdot p \quad (60)$$

as both  $v$  and  $p$  are constant. Hamilton's equations then transform as

$$\frac{d\tilde{q}(t)}{dt} = \frac{dq(t)}{dt} + v = \frac{\tilde{H}(\tilde{p})}{\partial \tilde{p}} = \frac{\partial H(p)}{\partial p} + v, \quad \frac{d\tilde{p}(t)}{dt} = \frac{dp(t)}{dt} = -\frac{\partial \tilde{H}(\tilde{p})}{\partial \tilde{q}} = 0 \quad (61)$$

and so the tilde equations are equivalent to the untilde'd Hamilton equations (46).

When the equations have this particularly simple form, extended bodies that are constituted of multiple particles, such as a human being, cannot distinguish between the moving and the rest frame within the context of classical mechanics. This is important as it allows us to travel on uniformly moving trains and jets. It was for this reason that Galileo introduced this as a relativity principle to explain why the earth could indeed be moving around the sun while we have the Ptolemaic perception that it is stationary. But this is just a property of a very particular degenerate solution. We know that such degenerate solutions break the symmetry of general systems of equations. This leads to a strong relativity,  $\tilde{H} = H$  and not the relativity or symmetry of the general set of equations that has  $\tilde{H} = H \circ \varrho^{-1}$ . Yet we have raised these inertial states based on this property of a highly degenerate specific solution to an almost exalted position in physics. An elementary particle state simply does not distinguish between inertial and noninertial states; it does not distinguish the inertial state as having a very special status. It is just a degenerate solution. It is the form of the equations, not a specific solution that must be invariant under the group.

Of course Galilean relativity is a limit of special relativity. The Lorentz group contracts to the Euclidean group. Relativity is fundamentally concerned with the concept of simultaneity and the ordering of events by different observers in different physical states. Special relativity has the property that simultaneity is relative to

the inertial state of observer state characterized by  $v$ . It assumes, or rather, is silent about whether simultaneity is affected by the relative noninertial state characterized by  $f, r$ . The Minkowski metric

$$d\tau^2 = dt^2 - \frac{1}{c^2} dq^2. \quad (62)$$

contracts to the degenerate Newtonian time line element in the limit of small velocities relative to  $c$ .

$$\gamma^\circ = \lim_{c \rightarrow \infty} dt^2 \left( 1 - \frac{v^2}{c^2} \right) = dt^2. \quad (63)$$

Simultaneity in the Galilean relativity limit is independent of both the relative inertial and noninertial state and so we say that it is absolute.

General relativity locally has the same concept of simultaneity as special relativity. It shows that gravity can be understood as a curvature of a manifold with locally inertial frames, in which special relativity continues to apply, and therefore simultaneity depends only on the relative local inertial state. In a system where there is only gravity, there are only locally inertial states; all particles follow geodesics that are inertial trajectories in the curved manifold and neighboring locally inertial frames are related by the connection. The covariant derivative is relative to these locally inertial frames related by the connection. General relativity, like special relativity, is silent about simultaneity and the clocks of particles in noninertial states due to other forces, a simple example of which is an electron in a magnetic field.

Just as Galilean relativity, that singles out inertial frames, is the limit of special relativity, this simple theorem about Hamilton's mechanics is the first pointer as the limit of a relativity theory in which simultaneity depends on the relative inertial and noninertial state of the observer, characterized by the relative  $v, f, r$  [8],[9-10]. This theory has a nondegenerate orthogonal Born metric [11],[12] on extended phase space. This results in a relative simultaneity between any states, inertial or noninertial.

It may appear that a relativity on extended phase space is not compatible with quantum mechanics. The opposite is actually true; pure quantum mechanical considerations point to this noninertial relativity also. The Heisenberg commutation relations are the Hermitian representation of the Weyl-Heisenberg algebra corresponding to the unitary representations of the Weyl-Heisenberg group. We normally quantize a relativistic theory. But we can do the opposite; we can determine the quantum theory with the maximal symmetry preserving the Heisenberg commutation relations and then endow it with an orthogonal line element that is required for relativistic concepts of simultaneity, causality, inertial or noninertial and so forth. That is, *relativize* the quantum theory. This leads us to the same Born metric on extended phase space that is no less physical than spacetime, but is nonabelian. One may think that it will result in wave functions that are functions of all the degrees of freedom of phase space. But it does not, the nonabelian Mackey theorems [13] results in wavefunctions of commuting subsets that we expect, such as space-time  $\psi(t, q)$  or energy-momentum  $\psi(e, p)$ . It also admits wave functions of momentum-time  $\psi(t, p)$  or energy-space  $\psi(e, q)$  that we do not generally consider but manifest this more general symmetry. This theory contracts in its ultimate limit to Hamilton's equations with the  $\mathcal{HSp}(2n)$  symmetry.

This starts with Dirac's observation[14] that the quantum phase is the most fundamental attribute of quantum mechanics. Therefore, physical states are rays in the Hilbert space that equivalence classes of states in the Hilbert space up to a phase. Quantum symmetries are projective representations of the group that are also defined up to a phase. A mathematical theorem states that projective representations are equivalence classes of ordinary unitary (or anti-unitary) representations of the central extension of the group [15-16]. The Weyl-Heisenberg group is a

particular one parameter central extension of the abelian group of translations on extended phase space. Therefore, the unitary representations of the Weyl-Heisenberg group are equivalent to a particular projective representation of the abelian group of translations on extended phase space. Quantum mechanics has lead us again to extended phase space.

The maximal group that preserves the Weyl-Heisenberg group and algebra is its automorphism group. This group is the central extension of a conformally scaled inhomogeneous symplectic group [5,10,17]. The symplectic subgroup of this group is precisely the symplectic group that appears in the theorem that we have established. The projective representations of this group, that are equivalence classes of the ordinary unitary representations of the automorphism group, defines the largest Hilbert space and quantum symmetries that preserve the Heisenberg commutation relations. The unitary representations that are determined by the nonabelian Mackey theorems [13] result in a Hilbert space with wave functions that are functions of commuting subsets as described above.

Therefore, quantum mechanics has again resulted in a formulation on extended phase space. Projective representations of abelian translations on extended phase space are equivalence classes of unitary representations of its central extension, the Weyl-Heisenberg group. We can now *relativize* the quantum theory (the opposite of the usual quantization of a relativistic theory) and we are lead to endow extended phase space with a Born metric [11],[12] This results in quantum relativistic theory that includes noninertial states with the more general notion of simultaneity depending on the general state of the particle. In the limit of small interactions that are *nearly* inertial, the special relativist theory is recovered [18] and in the ultimate limit of small interactions and velocities and  $\hbar \rightarrow 0$  contracts to the Hamilton symmetry.

On the other hand, special relativistic quantum mechanics is the projective representations of the inhomogeneous Lorentz group [19],[20] on spacetime and the Hilbert space consists only of inertial states. This considers only "half" of the extended phase space and does not address noninertial states.

The general noninertial symmetry of Hamilton's equations described in this paper points to this other half, a relativity where simultaneity depends also on the relative noninertial state of the particle [10]. Whether nature complies with this more general noninertial relativity is a theoretical and empirical question that must be investigated, unless, like Mrs S. in Sacks' classic [21], our inertial spacetime viewpoint prevents us from even conceiving of this possibility.

This paper is dedicated to Professor DeWitt-Morette for her lifelong dedication to understanding the interplay between mathematics and physics and giving an appreciation of that interplay to her students [22]. I would like to thank Peter Jarvis for discussions that have improved the clarity of these ideas.

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