

NONINERTIAL SYMMETRY OF HAMILTON'S MECHANICS

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ABSTRACT. We present a new derivation of Hamilton's equations that shows that they have a symmetry group $Sp(2n) \otimes_s \mathcal{H}(n)$. The group $Sp(2n)$ is the real noncompact symplectic group and $\mathcal{H}(n)$ is mathematically a Weyl-Heisenberg group that is parameterized by velocity, force and power where power is the central element of the group. The homogeneous Galilei group $\mathcal{E}(n) \simeq \mathcal{SO}(n) \otimes_s \mathcal{A}(n)$, where the special orthogonal group $\mathcal{SO}(n) \subset Sp(2n)$ is parameterized by rotations and the abelian group $\mathcal{A}(n) \subset \mathcal{H}(n)$ is parameterized by velocity, is the inertial subgroup.

1. SYMMETRY GROUP THEOREM OF HAMILTON'S EQUATIONS

Let $\mathbb{P} = \mathbb{R}^{2n+2}$ be an extended phase space with coordinates $\{z^a\} = \{y^\alpha, e, t\}$ where $a, b = 1, \dots, 2n+2$ and $\alpha, \beta = 1, \dots, 2n$. The $2n$ y -coordinates may also be written $\{y^\alpha\} = \{p^i, q^i\}$ with $i, j = 1, \dots, n$. In these coordinates, there is a symplectic metric that may be written in the forms

$$\omega = \zeta_{a,b} dz^a dz^b = \zeta^\circ_{\alpha,\beta} dy^\alpha dy^\beta - de \wedge dt = \delta_{i,j} dp^i \wedge dq^j - de \wedge dt. \quad (1)$$

The $2n+2$ dimensional square matrix of components $\zeta = [\zeta_{a,b}]$ is given by

$$\zeta = \begin{pmatrix} \zeta^\circ & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \zeta^\circ = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}, \quad (2)$$

and 1_n is the unit n dimensional square matrix. Assume also that there is a degenerate orthogonal line element

$$\gamma^\circ = dt^2 = \eta^\circ_{a,b} dz^a dz^b, \quad (3)$$

where the $\eta^\circ_{a,b}$ are the components of the $2n+2$ dimensional square matrix that is zero except for a 1 in the lower right hand corner,

$$\eta^\circ = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (4)$$

As $\mathbb{P} = \mathbb{R}^{2n+2}$, the coordinates and the form of the symplectic metric (2) and degenerate orthogonal line element (4) are defined globally.

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1.1. **Theorem.** Let \mathbb{P} be extended phase space as defined above with symplectic metric ω given in (1) and degenerate orthogonal line element γ° given in (3). Let ρ be a diffeomorphism $\rho : \mathbb{P} \rightarrow \mathbb{P} : z \mapsto \tilde{z} = \rho(z)$ that leaves invariant the symplectic metric, $\omega = \rho^*\omega$ and the degenerate orthogonal line element, $\gamma^\circ = \rho^*\gamma^\circ$. Then,
A) the connected group of transformations on the cotangent space leaving the symplectic metric and degenerate orthogonal line element invariant is

$$\mathcal{HSp}(2n) \simeq \mathcal{Sp}(2n) \otimes_s \mathcal{H}(n), \quad (5)$$

where $\mathcal{H}(n)$ is the Weyl-Heisenberg group and $\mathcal{Sp}(2n)$ is the real noncompact symplectic group¹[1].

B) locally the diffeomorphisms ρ must have Jacobian matrices that are an element of $\mathcal{HSp}(2n)$,

$$\left[\frac{\partial \rho^a(z)}{\partial z^b} \right] = \Gamma(z) \in \mathcal{HSp}(2n) \quad \forall z \in \mathbb{P}, \quad (6)$$

and consequently have a particular functional form that satisfy a first order set of differential equations that are Hamilton's equations [2].

1.2. **Comments.** In coordinates, the metric and line element pull back under the mapping $\tilde{z}^a = \rho^a(z)$ is

$$\begin{aligned} \omega &= \zeta_{a,b} d\tilde{z}^a d\tilde{z}^b = \zeta_{a,b} \frac{\partial \rho^a(z)}{\partial z^c} \frac{\partial \rho^b(z)}{\partial z^d} dz^c dz^d \\ \gamma^\circ &= \eta^\circ_{a,b} d\tilde{z}^a d\tilde{z}^b = \eta^\circ_{a,b} \frac{\partial \rho^a(z)}{\partial z^c} \frac{\partial \rho^b(z)}{\partial z^d} dz^c dz^d \end{aligned}$$

and so for the metric and line element to be invariant, the Jacobian matrices must satisfy

$$\zeta_{c,d} = \zeta_{a,b} \frac{\partial \rho^a(z)}{\partial z^c} \frac{\partial \rho^b(z)}{\partial z^d} \quad (7)$$

$$\eta^\circ_{c,d} = \eta^\circ_{a,b} \frac{\partial \rho^a(z)}{\partial z^c} \frac{\partial \rho^b(z)}{\partial z^d} \quad (8)$$

The proof that follows first shows that the matrix $\Gamma(z)$ that is defined in (6) and that satisfies these equations is an element of $\mathcal{HSp}(2n)$ and then that (6) is Hamilton's equations.

1.3. **Proof of Part A: Symmetry group is $\mathcal{HSp}(2n)$.** The symplectic metric on extended phase space is invariant under the symplectic group $\mathcal{Sp}(2n+2)$ and the degenerate orthogonal line dt^2 element is invariant under the affine group

$$\mathcal{IGL}(2n+1, \mathbb{R}) \simeq \mathcal{GL}(2n+1, \mathbb{R}) \otimes_s \mathcal{A}(2n+1), \quad \mathcal{A}(m) \simeq (\mathbb{R}^m, +). \quad (9)$$

We show in this section that the connected group that leaves both the symplectic metric ω and the degenerate orthogonal metric γ° is

$$\mathcal{HSp}(2n) \simeq \mathcal{Sp}(2n+2) \cap \mathcal{IGL}(2n+1, \mathbb{R}). \quad (10)$$

The symplectic metric ω given in (2) and degenerate orthogonal line element γ° given in (4) may be written in matrix notation as

$$\omega = dz^t \zeta dz, \quad dt^2 = dz^t \eta^\circ dz, \quad (11)$$

¹The notation varies from author to author, this group is often written as $\mathcal{Sp}(2n, \mathbb{R})$.

Using matrix notation, a transformation of the basis is $d\tilde{z} = \Gamma dz$, $\Gamma \in \mathcal{GL}(2n+2, \mathbb{R})$. It leaves invariant the symplectic metric if

$$\Gamma^t \zeta \Gamma = \zeta, \quad (12)$$

and the degenerate orthogonal line element is invariant if

$$\Gamma^t \eta^\circ \Gamma = \eta^\circ. \quad (13)$$

Expand the $2n + 2$ square matrix Γ as

$$\Gamma = \begin{pmatrix} \Sigma & b & w \\ c & a & r \\ d & g & \epsilon \end{pmatrix}, \quad (14)$$

where Σ is a $2n$ dimensional square matrix, $b, w \in \mathbb{R}^{2n}$ are column vectors, $c, d \in \mathbb{R}^{2n}$ are row vectors and $a, r, g, \epsilon \in \mathbb{R}$. Then expanding the expression for the invariance of the η° , (13), [3]

$$\begin{aligned} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} &= \begin{pmatrix} \Sigma^t & c^t & d^t \\ b^t & a & g \\ w^t & r & \epsilon \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \Sigma & b & w \\ c & a & r \\ d & g & \epsilon \end{pmatrix} \\ &= \begin{pmatrix} d^t d & d^t g & d^t \epsilon \\ g d & g^2 & g \epsilon \\ d \epsilon & g \epsilon & \epsilon^2 \end{pmatrix}. \end{aligned} \quad (15)$$

This identity requires $d = g = 0$ and $\epsilon = \pm 1$. Applying this to (14), and computing the determinant

$$\text{Det} \Gamma = \text{Det} \begin{pmatrix} \Sigma & b & w \\ c & a & r \\ 0 & 0 & \epsilon \end{pmatrix} = \epsilon \text{Det} \begin{pmatrix} \Sigma & b \\ c & a \end{pmatrix} \neq 0,$$

and so $\begin{pmatrix} \Sigma & b \\ c & a \end{pmatrix} \in \mathcal{GL}(2n+1, \mathbb{R})$ with $(w, r) \in \mathbb{R}^{2n+1}$.

A group \mathcal{G} is a semidirect product if it has a subgroup $\mathcal{K} \subset \mathcal{G}$ and a normal subgroup $\mathcal{N} \subset \mathcal{G}$ such that $\mathcal{G} \simeq \mathcal{N}\mathcal{K}$ and $\mathcal{K} \cap \mathcal{N} = \mathbf{e}$ where \mathbf{e} is the trivial group. It is straightforward to verify that the above matrices define the extended affine group

$$\mathcal{IGL}(2n+1, \mathbb{R}) \simeq \mathbb{Z}_2 \otimes_s \mathcal{IGL}(2n+1, \mathbb{R}), \quad (16)$$

where the affine group is

$$\mathcal{IGL}(2n+1, \mathbb{R}) \simeq \mathcal{GL}(2n+1, \mathbb{R}) \otimes_s \mathcal{A}(n+1). \quad (17)$$

The \mathbb{Z}_2 group, parameterized by $\epsilon = \pm 1$ is the discrete group that changes the sign of t . The affine group is the maximal connected subgroup. As we only require the connected component, we can set $\epsilon = 1$.

Next, the symplectic invariance condition (12) requires that

$$\begin{aligned} \begin{pmatrix} \zeta^\circ & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} &= \begin{pmatrix} \Sigma^t & c^t & 0 \\ b^t & a & 0 \\ w^t & r & 1 \end{pmatrix} \begin{pmatrix} \zeta^\circ & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \Sigma & b & w \\ c & a & r \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \Sigma^t \zeta^\circ \Sigma & \Sigma^t \zeta^\circ b & -c^t + \Sigma^t \zeta^\circ w \\ b^t \zeta^\circ \Sigma & 0 & -a + b^t \zeta^\circ w \\ c + w^t \zeta^\circ \Sigma & a + w^t \zeta^\circ b & 0 \end{pmatrix}. \end{aligned} \quad (18)$$

This identity is satisfied with

$$b = 0, a = 1, c = -w^t \zeta^\circ \Sigma, \quad \Sigma^t \zeta^\circ \Sigma = \zeta^\circ. \quad (19)$$

Γ now has the form

$$\Gamma(\Sigma, w, r) = \begin{pmatrix} \Sigma & 0 & w \\ -w^t \zeta^\circ A & 1 & r \\ 0 & 0 & 1 \end{pmatrix}. \quad (20)$$

where $\Sigma \in \mathcal{S}p(2n)$, $w \in \mathbb{R}^{2n}$ and $r \in \mathbb{R}$.

The group multiplication of the matrix group given by (20) is determined by matrix multiplication to be

$$\Gamma(\Sigma'', w'', r'') = \Gamma(\Sigma', w', r') \Gamma(\Sigma, w, r), \quad (21)$$

where

$$\begin{aligned} \Sigma'' &= \Sigma' \Sigma, \\ w'' &= w' + \Sigma' w, \\ r'' &= r' + r - w'^t \zeta^\circ \Sigma' w. \end{aligned} \quad (22)$$

and the inverse is determined by the matrix inverse to be

$$\Gamma^{-1}(\Sigma, w, r) = \Gamma(\Sigma^{-1}, -\Sigma^{-1} w, -r). \quad (23)$$

The following groups are subgroups

$$\begin{aligned} \Gamma(\Sigma, 0, 0) &\in \mathcal{S}p(2n), \\ \Gamma(1_{2n}, w, r) &= \Upsilon(w, r) \in \mathcal{H}(n) \simeq \mathcal{A}(n) \otimes_s \mathcal{A}(n+1). \end{aligned} \quad (24)$$

where $\mathcal{A}(m)$ is the real abelian group under addition, $\mathcal{A}(m) \simeq (\mathbb{R}^m, +)$. It is then be shown that $\Upsilon(w, r) \in \mathcal{H}(n)$ [4] is a normal subgroup by computing the automorphisms

$$\begin{aligned} \Upsilon(w'', r'') &= \Gamma(\Sigma', w', r') \Upsilon(w, r) \Gamma^{-1}(\Sigma', w', r') \\ &= \Upsilon(\Sigma' w, r + (\Sigma' w)^t \zeta^\circ w' - w'^t \zeta^\circ \Sigma' w). \end{aligned} \quad (25)$$

As

$$\frac{\Gamma(1_{2n}, w, r) \cap \Gamma(\Sigma, 0, 0)}{\Gamma(\Sigma, w, r)} = \frac{\Gamma(1_{2n}, 0, 0)}{\Gamma(\Sigma, 0, 0)}, \quad (26)$$

it follows that the intersection of the groups is the identity and $\mathcal{H}Sp(2n) \simeq \mathcal{H}(n)Sp(2n)$ group is the semidirect product (5) as claimed.

It is straightforward to show with $\epsilon = \pm 1$ that the intersection of the symplectic and extended affine group is

$$\mathcal{S}p(2n+2) \cap \mathcal{I}\hat{\mathcal{G}}\mathcal{L}(2n+1, \mathbb{R}) \simeq \mathcal{H}\hat{\mathcal{S}}p(2n), \quad \mathcal{H}\hat{\mathcal{S}}p(2n) \simeq \mathbb{Z}_2 \otimes_s \mathcal{H}Sp(2n) \quad (27)$$

where again the \mathbb{Z}_2 changes the sign of t .

That $\mathcal{H}(n)$ is the Weyl-Heisenberg group may be determined by computing its algebra

$$W_\alpha = \frac{\partial}{\partial w^\alpha} \Upsilon(w, r)|_{w=r=0}, \quad U = \frac{\partial}{\partial r} \Upsilon(w, r)|_{w=r=0}. \quad (28)$$

A general element of the algebra is $Z = w^\alpha W_\alpha + rU$. The Lie algebra of a matrix group is the matrix commutators $[A, B] = AB - BA$ that give

$$[W_\alpha, W_\beta] = 2\zeta^\circ_{\alpha, \beta} U, \quad [W_\alpha, U] = 0. \quad (29)$$

This is the Weyl-Heisenberg algebra where U is the central generator. The factor of 2 is just normalization. It can be removed simply by scaling $r \mapsto 2r$.

This completes the proof of Part A of the theorem that establishes that the connected group that has both symplectic and affine symmetry is $\mathcal{HS}p(2n)$.

1.4. Proof of Part B: Diffeomorphisms satisfy Hamilton's equations.

The Jacobian matrix $[\frac{\partial \rho(z)}{\partial z}]$ of the diffeomorphism ρ that leaves invariant the symplectic metric (3) and the degenerate orthogonal line element (1) must satisfy (12) and (13). Therefore, the Jacobian matrix is an element of the symmetry group, $[\frac{\partial \rho(z)}{\partial z}] = \Gamma(z) \in \mathcal{HS}p(n)$. Using the expanded notation with $\{z\} = \{y, e, t\}$ with $z \in \mathbb{R}^{2n+2}$, $y \in \mathbb{P}^\circ \simeq \mathbb{R}^{2n}$ and $e, t \in \mathbb{R}$, and likewise for the diffeomorphism

$$\{\rho(z)\} = \{\rho_y(y, e, t), \rho_e(y, e, t), \rho_t(y, e, t)\} \quad (30)$$

the Jacobian matrix is

$$\begin{pmatrix} \frac{\partial \rho_y(y, e, t)}{\partial y} & \frac{\partial \rho_y(y, e, t)}{\partial e} & \frac{\partial \rho_y(y, e, t)}{\partial t} \\ \frac{\partial \rho_e(y, e, t)}{\partial y} & \frac{\partial \rho_e(y, e, t)}{\partial e} & \frac{\partial \rho_e(y, e, t)}{\partial t} \\ \frac{\partial \rho_t(y, e, t)}{\partial y} & \frac{\partial \rho_t(y, e, t)}{\partial e} & \frac{\partial \rho_t(y, e, t)}{\partial t} \end{pmatrix} = \begin{pmatrix} \Sigma(z) & 0 & w(z) \\ -w^t(z)\zeta^\circ & \Sigma(z) & 1 & r(z) \\ 0 & 0 & 1 \end{pmatrix}. \quad (31)$$

where we are suppressing indices and using matrix notation.

This restricts the functional dependency of the diffeomorphisms as follows. First the time component, $\frac{\partial \rho_t(y, e, t)}{\partial y} = \frac{\partial \rho_t(y, e, t)}{\partial e} = 0$ and $\frac{\partial \rho_t(y, e, t)}{\partial t} = 1$ and so ignoring trivial integration constants, $\rho_t(y, e, t) = t$. Next for the energy component, note that $\frac{\partial \rho_e(y, e, t)}{\partial e} = 1$ and therefore ρ_e may be written as $\rho_e(y, e, t) = e + H(y, t)$ where H is some function. Finally, $\frac{\partial \rho_y(y, e, t)}{\partial e} = 0$ and consequently $\rho_y(y, e, t) = \varphi(y, t)$ where φ is some function.

Summarizing, the diffeomorphism $\tilde{z} = \rho(z)$ can be expanded as

$$\begin{aligned} \tilde{y} &= \rho_y(y, e, t) = \varphi(y, t) = \phi_y(t), \\ \tilde{e} &= \rho_e(y, e, t) = e + H(y, t), \\ \tilde{t} &= \rho_t(y, e, t) = t. \end{aligned} \quad (32)$$

H and φ are functions

$$\begin{aligned} H &: \mathbb{R}^{2n+1} \rightarrow \mathbb{R} : (y, t) \mapsto H(y, t), \\ \varphi &: \mathbb{R}^{2n+1} \rightarrow \mathbb{P}^\circ : (y, t) \mapsto \varphi(y, t). \end{aligned} \quad (33)$$

ϕ_y are the curves defined by

$$\phi_y : \mathbb{R} \rightarrow \mathbb{P}^\circ : t \mapsto \phi_y(t) = \varphi(y, t), \quad \phi_y(0) = \varphi(y, 0) = y. \quad (34)$$

H will turn out to be the Hamiltonian and ϕ_y the curves that are the trajectories in phase space that are solutions to Hamilton's equations.

Substituting these back into (31), the Jacobian now has the form

$$\begin{pmatrix} \frac{\partial \varphi(y, t)}{\partial y} & 0 & \frac{\partial \varphi(y, t)}{\partial t} \\ \frac{\partial H(y, t)}{\partial y} & 1 & \frac{\partial H(y, t)}{\partial t} \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \Sigma(y, t) & 0 & w(y, t) \\ -w^t(y, t)\zeta^\circ & \Sigma(y, t) & 1 & r(y, t) \\ 0 & 0 & 1 \end{pmatrix}. \quad (35)$$

Therefore we have

$$\frac{\partial \varphi(y, t)}{\partial y} = \Sigma(y, t), \quad \frac{\partial H(y, t)}{\partial y} = -\left[\frac{\partial \varphi(y, t)}{\partial t}\right]^t \zeta^\circ \Sigma(y, t), \quad \frac{\partial H(y, t)}{\partial t} = r(y, t) \quad (36)$$

As $\varphi(y, t)$ is a canonical transformation for some y° , $y = \varphi(y^\circ, t)$ and for some t° , $y^\circ = \phi_{y^\circ}(t^\circ)$ with $\Sigma(y^\circ, t^\circ) = 1_{2n}$. Then from the chain rule,

$$\frac{\partial \varphi(y, t)}{\partial t} = \frac{\partial \varphi(y, t)}{\partial y} \frac{\partial \varphi(y^\circ, t)}{\partial t} = \Sigma(y, t) \frac{\partial \varphi(y^\circ, t)}{\partial t} = \Sigma(y, t) \frac{d\phi_{y^\circ}(t)}{dt} \quad (37)$$

Consequently

$$\frac{\partial H(y, t)}{\partial y} = -\left[\frac{d\phi_{y^\circ}(t)}{dt}\right]^\dagger \Sigma^\dagger(y, t) \zeta^\circ \quad \Sigma(y, t) = -\left[\frac{d\phi_{y^\circ}(t)}{dt}\right]^\dagger \zeta^\circ \quad (38)$$

Re-arranging

$$\frac{d\phi_{y^\circ}(t)}{dt} = -\zeta^\circ \left[\frac{\partial H(y, t)}{\partial y}\right]^\dagger, \quad \frac{\partial H(y, t)}{\partial t} = r(y, t) \quad (39)$$

In component form this is

$$\frac{d\phi_{y^\circ}^\alpha(t)}{dt} = \zeta^{\circ\alpha, \beta} \frac{\partial H(y, t)}{\partial y^\beta}, \quad \frac{\partial H(y, t)}{\partial t} = r(y, t) \quad (40)$$

where $[\zeta^{\circ\alpha, \beta}] = -\zeta^\circ$. These are Hamilton's equations with the initial point $y^\circ = \phi_{y^\circ}(t^\circ)$.

The converse requires us to prove that if the diffeomorphisms satisfy Hamilton's equations (40), then the symplectic and line element are invariant.

$$\begin{aligned} \tilde{\omega} &= d\tilde{y}^\dagger \zeta^\circ d\tilde{y} + d\tilde{t} \wedge d\tilde{e} \\ &= (dy + d\phi_{y^\circ}(t))^\dagger \zeta^\circ (dy + d\phi_{y^\circ}(t)) + dt \wedge (de + dH(y, t)) \\ &= dy^\dagger \zeta^\circ dy + dt \wedge de - \left[\frac{d\phi_{y^\circ}(t)}{dt}\right]^\dagger \zeta^\circ dy \wedge dt - \frac{\partial H(y, t)}{\partial y} dy \wedge dt \\ &= \omega - \left(\left[\zeta^\circ \frac{d\phi_{y^\circ}(t)}{dt}\right]^\dagger - \frac{\partial H(y, t)}{\partial y} \right) dy \wedge dt \\ &= \omega \end{aligned} \quad (41)$$

$\gamma^\circ = dt^2$ is invariant as t is an invariant parameter in Hamilton's equations. This completes the proof of the theorem.

A corollary of the theorem is that Hamilton's equations are valid in any extended canonical coordinates where the symplectic metric and degenerate line element have the form given in (1) and (3). Furthermore, transformations between these extended canonical coordinates must have a Jacobian that is an element of the $\mathcal{HSp}(2n)$ group (6).

2. PHYSICAL MEANING OF THE THEOREM

The symplectic symmetry and affine symmetries are very well known to be fundamental symmetries of classical mechanics. It should not therefore be a surprise that the intersection of these symmetries, where both are manifest, plays a fundamental role in Hamilton's mechanics.

An element $\Gamma \in \mathcal{HSp}(2n) \simeq \mathcal{Sp}(2n) \otimes_s \mathcal{H}(n)$, due to the defining properties of the semidirect product can always be written as the product of a symplectic transformation and a Weyl-Heisenberg transformation

$$\Gamma(\Sigma, y, r) = \Gamma(1_n, y, r)\Gamma(\Sigma, 0, 0). \quad (42)$$

We will consider the symplectic group first and show that this is the standard canonical transforms on phase space. Next, we consider the Weyl-Heisenberg transformations and show that they lead to familiar results.

2.1. Symplectic transformations. Consider first the symplectic transformations. In this case, the general transformations (32) reduce to

$$\tilde{y} = \rho_y(y, t) = \varphi(y, t), \quad \tilde{e} = \rho_e(e) = e, \quad \tilde{t} = \rho_t(t) = t, \quad (43)$$

with Jacobian satisfying

$$d\tilde{y} = \frac{\partial \varphi(y, t)}{\partial y} dy = \Sigma(y, t) dy. \quad (44)$$

The $\varphi(y, t)$ are time dependent canonical transformations that appear in all the standard treatments of Hamilton's mechanics. They may be regarded as the canonical transformations parameterized by time on the momentum, position phase space $y \in \mathbb{P}^\circ \simeq \mathbb{R}^{2n}$

$$\varphi_t : \mathbb{P}^\circ \rightarrow \mathbb{P}^\circ : y \mapsto \tilde{y} = \varphi_t(y), \quad (45)$$

or as the curves $\phi_y : \mathbb{R} \rightarrow \mathbb{P}^\circ$ that are given in (34). The solutions ϕ_y to Hamilton's equations may be regarded as a time evolving canonical transformation.

The coordinates in which the symplectic metric have the canonical form (12) are canonical coordinates. In particular, Hamilton's equations are valid in any canonical coordinates $\tilde{y} = \varrho(y)$ with

$$d\tilde{y} = \frac{\partial \varrho(y)}{\partial y} dy = \Sigma(y) dy. \quad (46)$$

Hamilton's equations in the tilde coordinates are

$$\frac{d\tilde{\phi}_{y^\circ}(t)}{dt} = -\zeta^\circ \left[\frac{\partial \tilde{H}(\tilde{y}, t)}{\partial \tilde{y}} \right]^\dagger, \quad (47)$$

with

$$\tilde{H}(\tilde{y}, t) = \tilde{H}(\varrho(y), t) = H(y, t), \quad \tilde{\phi}_{y^\circ}(t) = \varrho(\phi_{y^\circ}(t)), \quad (48)$$

and therefore

$$\tilde{H} = H \circ \varrho^{-1} \text{ and } \tilde{\phi}_{y^\circ} = \varrho \circ \phi_{y^\circ}. \quad (49)$$

It then follows from the methods used to prove the general theorem that Hamilton's equations transform into the non-tilde coordinates for the transforms ϱ that are the time independent special case of the more general ρ transforms of the theorem.

Note particularly that under a canonical transformation, that the Hamiltonian transforms as $\tilde{H} = H \circ \varrho^{-1}$ given in (49) and not as an invariant function $\tilde{H} = H$. Canonical coordinates do not have the concept of states being inertial or noninertial and Hamilton's equations are valid in either provided that the Hamiltonian $H(y, t)$ is chosen appropriately according to (49).

The phase space \mathbb{P}° may be generalized to symplectic manifolds with Hamilton's equations expressed as the flows of Hamiltonian vector fields [5].

2.2. Weyl-Heisenberg transformations. Define $y = (p, q)$, $p, q \in \mathbb{R}^n$ and $\phi = (\pi, \xi)$. In components, this is $\{y^a\} = \{p^i, q^i\}$, $\{\phi^a(t)\} = \{\pi^i(t), \xi^i(t)\}$ $i, j = 1, \dots, n$. As is usual, p is canonical momentum and q is canonical position. We will continue to use matrix notation with indices suppressed. Hamilton's equations then take on their most simple form,

$$\frac{d\xi(t)}{dt} = v = \frac{\partial H(p, q, t)}{\partial p}, \quad \frac{d\pi(t)}{dt} = f = -\frac{\partial H(p, q, t)}{\partial q}, \quad \frac{\partial H(p, q, t)}{\partial t} = r, \quad (50)$$

where $v(p, q, t), f(p, q, t) \in \mathbb{R}^n$ are the velocity and force respectively and $r(p, q, t) \in \mathbb{R}$ is the power. The velocity force and power are generally functions of (p, q, t) and this will be implicit in the following. The Weyl-Heisenberg subgroup may be written as

$$\Upsilon(f, v, r) = \Gamma(1_{2n}, f, v, r) = \begin{pmatrix} 1_n & 0 & 0 & f \\ 0 & 1_n & 1 & v \\ v & -f & 1 & r \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (51)$$

The coordinates z of the extended phase space \mathbb{P} may be similarly expanded as $z = (p, q, e, t)$ and the Weyl-Heisenberg transformation $d\tilde{z} = \Upsilon dz$ expands as

$$\begin{pmatrix} d\tilde{p} \\ d\tilde{q} \\ d\tilde{e} \\ d\tilde{t} \end{pmatrix} = \begin{pmatrix} 1_n & 0 & 0 & f \\ 0 & 1_n & 1 & v \\ v & -f & 1 & r \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} dp \\ dq \\ de \\ dt \end{pmatrix}. \quad (52)$$

Using Hamilton's equations (50), this results in

$$\begin{aligned} d\tilde{t} &= dt, \\ d\tilde{q} &= dq + vdt && = dq + d\xi(t), \\ d\tilde{p} &= dp + fdt && = dp + d\pi(t), \\ d\tilde{e} &= de + v \cdot dp - f \cdot dq + rdt && = de + dH(p, q, t). \end{aligned} \quad (53)$$

These are the transformations that relate two states in extended phase space that have a relative rate of change of position, momentum and energy with respect to time. That is, they have a relative velocity v , force f and power r . These are general states in the extended phase space that may be inertial or noninertial. In the energy transformation, $\int v \cdot dp$ is the incremental kinetic energy and $-\int f \cdot dq$ is the work transforming from energy state e to \tilde{e} . The term $\int rdt$ is the explicit power for time dependent Hamiltonians. Solving Hamilton's equations enables these to be integrated to the form that is a special case of (32) with $\Sigma = 1_{2n}$,

$$\begin{aligned} \tilde{t} &= \rho_t(t) = t, \\ \tilde{q} &= \rho_q(q, t) = q + \xi(t), \\ \tilde{p} &= \rho_p(p, t) = p + \pi(t), \\ \tilde{e} &= \rho_e(e, p, q, t) = e + H(p, q, t). \end{aligned} \quad (54)$$

Using the group multiplication (21-22) with $\Sigma = 1_{2n}$, or simply multiplying the matrices in (52) together shows that

$$\Upsilon(\tilde{f}, \tilde{v}, \tilde{r})\Upsilon(f, v, r) = \Upsilon(f + \tilde{f}, v + \tilde{v}, r + \tilde{f}v - \tilde{v}f), \quad (55)$$

$$\Upsilon(f, v, r)\Upsilon(\tilde{f}, \tilde{v}, \tilde{r}) = \Upsilon(f + \tilde{f}, v + \tilde{v}, r - \tilde{f}v + \tilde{v}f). \quad (56)$$

These are not equal and consequently the operations do not commute. This can be made even more explicit by considering the case of a transformation in velocity followed by a transformation in force

$$\Upsilon(\tilde{f}, 0, 0)\Upsilon(0, v, 0) = \Upsilon(\tilde{f}, v, \tilde{f}v) \quad (57)$$

$$\Upsilon(0, v, 0)\Upsilon(\tilde{f}, 0, 0) = \Upsilon(\tilde{f}, v, -\tilde{f}v) \quad (58)$$

This is not unexpected. We do not expect an inertial transformation in velocity followed by a noninertial transformation in force to be the same as the noninertial force transformation followed by the inertial velocity transformation. What is

unexpected is that the noncommutivity is given precisely by the Weyl-Heisenberg nonabelian group. The noncommutativity is also why noninertial states and frames are difficult to work with.

3. DISCUSSION

Hamilton's mechanics is a reformulation of Newton's mechanics and is therefore invariant under Galilean relativity. The homogeneous Galilei relativity group is mathematically the Euclidean group $\mathcal{E}(n) \simeq \mathcal{SO}(n) \otimes_s \mathcal{A}(n)$ parameterized by rotations and velocity. This is a subgroup of the group of transformations $\mathcal{HSp}(2n)$. The orthogonal group $\mathcal{SO}(n) \subset \mathcal{Sp}(2n)$ where in this case the symplectic transformations on \mathbb{P}° are just the rotations

$$\Sigma(R) = \begin{pmatrix} R & 0 \\ 0 & R \end{pmatrix}. \quad (59)$$

The space time translations are a subgroup of the Weyl-Heisenberg group, $\mathcal{A}(n) \subset \mathcal{H}(n) \simeq \mathcal{A}(n) \otimes_s \mathcal{A}(n+1)$. The resulting transformations are the inertial transformations on extended phase space

$$\begin{aligned} d\tilde{t} &= dt, \\ d\tilde{q} &= Rdq + vdt, \\ d\tilde{p} &= Rdp, \\ d\tilde{e} &= de + v \cdot dp. \end{aligned} \quad (60)$$

But why select this particular special case of the general $\mathcal{HSp}(2n)$ symmetry and give it the elevated status of a relativity group?

Up to this point we have not made any comment on the particular functional form of the Hamiltonian $H(p, q, t)$. The theorem is silent on its form. Physical considerations lead to Hamiltonians of many forms. For nonrelativistic electrodynamic, it is

$$H(p, q, t) = \frac{1}{2m} \left(p - \frac{\epsilon}{c} A(q, t) \right)^2 + \epsilon \phi(q, t) \quad (61)$$

where in this equation $\phi(q, t)$ is the electric potential and ϵ is the charge. The canonical momentum is related to the velocity through the expression

$$v(p, q, t) = \frac{p}{m} - \frac{\epsilon}{mc} A(q, t) \quad (62)$$

and so the relationship between velocity and momentum may be quite complex

For a broad class of problems in elementary classical mechanics, the Hamiltonian is given simply by

$$H(p, q, t) = K(p) + V(q) = \frac{p^2}{2m} + V(q). \quad (63)$$

Hamilton's equations result in $v = \frac{p}{m}$ and $\int v \cdot dp = \frac{p^2}{2m}$ is the kinetic energy $K(p)$ and $-\int f \cdot dq = V(q)$ is the potential energy. Energy is constant in time as $\frac{\partial}{\partial t} H(p, q) = 0$. This is but a most basic solution. An even more basic case is the inertial state where $f = r = 0$ and therefore $V(q) = 0$. This state has the property that, from (53),

$$\tilde{H}(\tilde{p}) = H(p) + v \cdot p \quad (64)$$

as both v and p are constant. Hamilton's equations then transform as

$$\frac{d\tilde{q}(t)}{dt} = \frac{dq(t)}{dt} + v = \frac{\tilde{H}(\tilde{p})}{\partial\tilde{p}} = \frac{\partial H(p)}{\partial p} + v, \quad \frac{d\tilde{p}(t)}{dt} = \frac{dp(t)}{dt} = -\frac{\partial\tilde{H}(\tilde{p})}{\partial\tilde{q}} = 0 \quad (65)$$

and so the tilde equations are equivalent to the untilde'd Hamilton equations (50) with $\tilde{H} = H$ as functions .

When the equations have this particularly simple form, extended bodies that are constituted of multiple particles, such as a human being, cannot distinguish between the moving and the rest frame within the context of classical mechanics. This is important as it allows us to travel on uniformly moving trains and jets. It was for this reason that Galileo introduced this as a relativity principle to explain why the earth could indeed be moving around the sun while we have the Ptolemaic perception that it is stationary. But this is just a property of a very particular degenerate solution. We know that such degenerate solutions break the symmetry of general systems of equations. This leads to a strong relativity, $\tilde{H} = H$ and not the relativity or symmetry of the general set of equations that has $\tilde{H} = H \circ \varrho^{-1}$. Yet we have raised these inertial states based on this property of a highly degenerate specific solution to an almost exalted position in physics. An elementary particle state simply does not distinguish between inertial and noninertial states; it does not distinguish the inertial state as having a very special status. It is just a degenerate solution. It is the form of the equations, not a specific solution that must be invariant under the group.

Of course Galilean relativity is a limit of special relativity. The Lorentz group contracts to the Euclidean group. Relativity is fundamentally concerned with the concept of simultaneity and the ordering of events by different observers in different physical states. Special relativity has the property that simultaneity is relative to the inertial state of observer state characterized by v . It assumes, or rather, is silent about whether simultaneity is affected by the relative noninertial state characterized by f, r . The Minkowski metric

$$d\tau^2 = dt^2 - \frac{1}{c^2}dq^2. \quad (66)$$

contracts to the degenerate Newtonian time line element in the limit of small velocities relative to c .

$$\gamma^\circ = \lim_{c \rightarrow \infty} dt^2 \left(1 - \frac{v^2}{c^2}\right) = dt^2. \quad (67)$$

Simultaneity in the Galilean relativity limit is independent of both the relative inertial and noninertial state and so we say that it is absolute.

General relativity locally has the same concept of simultaneity as special relativity. It shows that gravity can be understood as a curvature of a manifold with locally inertial frames, in which special relativity continues to apply, and therefore simultaneity depends only on the relative local inertial state. In a system where there is only gravity, there are only locally inertial states; all particles follow geodesics that are inertial trajectories in the curved manifold and neighboring locally inertial frames are related by the connection. The covariant derivative is relative to these locally inertial frames related by the connection. General relativity, like special relativity, is silent about simultaneity and the clocks of particles in

noninertial states due to other forces, a simple example of which is an electron in a magnetic field.

Just as Galilean relativity, that singles out inertial frames, is the limit of special relativity, this simple theorem about Hamilton's mechanics is the first pointer as the limit, to a relativity theory in which simultaneity depends on the relative inertial and noninertial state of the observer, characterized by the relative v, f, r [2],[6] This theory has a nondegenerate orthogonal Born metric [7],[8] on extended phase space. This results in a relative simultaneity between any states, inertial or noninertial.

It may appear that a relativistic symmetry group on extended phase space is not compatible with quantum mechanics. The quantum symmetry is given by the projective representations that are equivalent to equivalence classes of unitary representations of the central extension of the group [9, 10].

Recall that the central extension of the inhomogeneous Euclidean group, $\mathcal{IE}(n) \simeq \mathcal{E}(n) \otimes_s \mathcal{A}(n+1)$, is the Galilei group

$$\mathcal{Ga}(n) = \overline{\mathcal{E}}(n) \otimes_s \mathcal{A}(n+1) \otimes_s \mathcal{A}(1).$$

The generator of the central $\mathcal{A}(1)$ subgroup is nonrelativistic mass that this group admits as an algebraic extension. The central extension of the inhomogeneous Hamilton group $\mathcal{IHa}(n) = \mathcal{Ha}(n) \otimes_s \mathcal{A}(2n)$ is

$$\mathcal{I\overline{H}a}(n) = \overline{\mathcal{Ha}}(n) \otimes_s \mathcal{H}(n+1) \otimes_s \mathcal{A}(2).$$

The Galilei group is the inertial subgroup of this group with mass one of the generators of the central $\mathcal{A}(2)$ subgroup. The Weyl-Heisenberg $\mathcal{H}(n+1)$ is parameterized by time, position, momentum and energy and the Hermitian representation of its algebra are the Heisenberg commutation relations. The projective representations of the inhomogeneous Hamilton group are equivalence classes of the unitary representations of this central extension. These may be computed using the Mackey theorems for unitary representations of semidirect product groups. One finds from this that the Hilbert space is of the form $\mathbf{H} \otimes \mathbf{L}^2(\mathbb{R}^{n+1}, \mathbb{C})$. Wave functions are of the form $\psi(q, t)$, or $\psi(p, t)$ as we expect and not wave functions of all the phase space degrees of " $\psi(t, q, p, e)$ ". This is also the case in the relativistic generalization [11],[12].

The theorem that shows that Hamilton's equations have the symmetry $Sp(2n) \otimes_s \mathcal{H}(n)$ should not be surprising as it is the intersection of a symplectic and affine symmetry, both of which are fundamental in classical mechanics. This does not give new results for classical mechanics but does give new insight into noninertial frames. There is no reason to single out inertial frames in Hamilton's mechanics as the equations are equally valid in inertial and noninertial states provided the appropriate Hamilton function is used. This does point to immediate relativistic [11], quantum [6] and quantum relativistic theories [12] were the noninertial symmetry in their context does have profound implication.

This paper is dedicated to Professor DeWitt-Morette for her lifelong dedication to understanding the interplay between mathematics and physics and giving an appreciation of that interplay to her students. I would like to thank Peter Jarvis for discussions that have improved the clarity of these ideas.

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