Projective Representations of the Hamilton Group: Noninertial symmetry in quantum mechanics

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Fundamental QM representation theorem

Physical states in quantum mechanics are rays Ψ.

Rays are the equivalence classes of states in a Hilbert space **H** defined up to a phase

$$|\psi\rangle \simeq |\tilde{\psi}\rangle \in \Psi \ \text{ iff } \ |\tilde{\psi}\rangle = e^{i\,\omega}\,|\psi\rangle, \qquad |\psi\rangle, |\tilde{\psi}\rangle \in \mathbf{H}, \ \ \omega \in \mathbb{R}.$$

Physical observables are the square of the modulus that is same for any representative of the ray

$$|\,(\Psi_\beta,\ \Psi_\alpha)|^2 = \quad |\,\langle\psi_\beta,\ \psi_\alpha\rangle|^2 = \left|\left\langle\tilde{\psi}_\beta,\ \tilde{\psi}_\alpha\right\rangle\right|^2$$

A projective representation of a symmetry Lie group $\mathcal{G}$ leaves invariant the square of the modulus

If $\tilde{\Psi}_\alpha = \varrho(g)\,\Psi_\alpha\ ,\ g \in \mathcal{G}$

$$|\,(\tilde{\Psi}_\beta,\ \tilde{\Psi}_\alpha)|^2 = |\,(\Psi_\beta,\ \Psi_\alpha)|^2,$$

Theorem (Bargmann, Mackey): A projective representation of a connected Lie group is equivalent to the ordinary unitary representations of its central extension

Central extensions

The central extension of a connected group $\mathcal{G}$ by the abelian group $\mathcal{Z}$ is the unique maximal group $\check{\mathcal{G}}$ satisfying the short exact sequence

$$e \to \mathcal{Z} \overset{\iota}{\to} \check{\mathcal{G}} \overset{\pi}{\to} \mathcal{G} \to e, \quad \mathcal{Z} \simeq \mathbb{A} \otimes \mathcal{A}(m).$$

$\mathbb{A}$ is a finite abelian group and $\mathcal{A}(m) \simeq (\mathbb{R}^m, +)$. The exact sequence decomposes into

$$e \to \mathbb{A} \to \overline{\mathcal{G}} \to \mathcal{G} \to e, \quad e \to \mathcal{A}(m) \to \check{\mathcal{G}} \to \overline{\mathcal{G}} \to e$$

where $\overline{\mathcal{G}}$ is the universal cover and $\mathbb{A}$ is the fundamental homotopy group.

As $\check{\mathcal{G}}$, $\overline{\mathcal{G}}$ are simply connected, they are characterized by their algebra.

Central extension of the algebra: Given a basis $\{X_a\}$ of the algebra of a group $\mathcal{G}$

$$[X_a, X_b] = c_{a,b}^c X_c$$

find maximal set of generators $\{X_a, A_i\}$

$$[X_a, X_b] = c_{a,b}^c X_c + c_{a,b}^i A_i, \quad [X_a, A_i] = 0, \quad [A_i, A_j] = 0$$

that satisfy the Jacobi identities where we discard trivial cases $X_a \mapsto X_a + A_a$.

Simply connected group for this algebra is $\check{\mathcal{G}}$.

Semidirect product theorems

Levi's Theorem: Any simply connected group $\mathcal{G}$ is equivalent to the semidirect product $\mathcal{G} \simeq \mathcal{K} \otimes_s \mathcal{N}$ where $\mathcal{K}$ is a semisimple group and $\mathcal{N}$ is a solvable normal group Levi's theorem always applies to a central extension $\check{\mathcal{G}}$ as it is simply connected.

Automorphism Theorem: A semidirect product $\mathcal{G} \simeq \mathcal{K} \otimes_s \mathcal{N}$ with $\mathcal{N}$ as a normal subgroup is homomorphic to a subgroup of $\mathcal{Aut}_{\mathcal{N}}$.

Mackey Semidirect Product Representation Theorems: Provide a prescription for determining the unitary irreducible representations of $\mathcal{G} = \mathcal{K} \otimes_s \mathcal{N}$ in terms of certain representations of the *Little* group $\mathcal{K}^\circ$ and stabilizer $\mathcal{G}^\circ \simeq \mathcal{K}^\circ \otimes_s \mathcal{N}$. Valid for non-abelian $\mathcal{N}$.

Special relativistic quantum mechanics

A well known example is the formulation of inertial states of special relativistic quantum mechanics as the projective representation of the inhomogeneous Lorentz group

$$\mathcal{I}\mathcal{L}(1, n) \simeq \mathcal{L}(1, n) \otimes_s \mathcal{A}(n+1), \quad \mathcal{A}(m) \simeq (\mathbb{R}^m, +)$$

It does not have an algebraic extension; central extension is the cover. For $n = 3$

$$\mathcal{P}(1, 3) \simeq \check{\mathcal{I}}\mathcal{L}(1, 3) \simeq \overline{\mathcal{L}}(1, 3) \otimes_s \mathcal{A}(4) \simeq \mathcal{S}\mathcal{L}(2, \mathbb{C}) \otimes_s \mathcal{A}(4)$$

Satisfies Levi's theorem: The semisimple group is $\mathcal{K} \simeq \mathcal{S}\mathcal{L}(2, \mathbb{C})$ and the solvable normal subgroup is $\mathcal{N} \simeq \mathcal{A}(4)$.

Unitary representations of Poincaré group determined using Mackey's theorems.

Nonrelativistic limit

In the ‘nonrelativistic’ $c \to \infty$ limit, $\mathcal{L}(1, n) \to \mathcal{E}(n) \simeq \mathcal{SO}(n) \otimes_s \mathcal{A}(n)$

Non-relativistic inertial state symmetry is

$$\mathcal{IE}(n) \simeq \mathcal{E}(n) \otimes_s \mathcal{A}(n+1)$$

Algebra is $Z = \alpha^{i,j} J_{i,j} + v^i G_i + q^i P_i + t E$

$$[J_{i,j}, J_{k,l}] = J_{j,k}\, \delta_{i,l} + J_{i,l}\, \delta_{j,k} - J_{i,k}\, \delta_{j,l} - J_{j,l}\, \delta_{i,k},$$

$$[J_{i,j}, G_k] = G_j\, \delta_{i,k} - G_i\, \delta_{j,k}, \quad [J_{i,j}, P_k] = P_i\, \delta_{j,k} - P_j\, \delta_{i,k},$$

$$[G_i, E] = P_i, \quad [G_i, P_k] = 0\,.$$

Admits algebraic central extension $[G_i, P_k] = M\, \delta_{i,k}$

Central extension is $\check{\mathcal{IE}}(n) = \overline{\mathcal{G}a}\,(n)$

$$\mathcal{G}a(n) \simeq \mathcal{E}(n) \otimes_s (\mathcal{A}(n+1) \otimes \mathcal{A}(1))$$

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$$[J_{i,j},\ G_k] = G_j\,\delta_{i,k} - G_i\,\delta_{j,k}, \quad [J_{i,j},\ P_k] = P_i\,\delta_{j,k} - P_j\,\delta_{i,k},$$

$$[G_i,\ E] = P_i, \quad [G_i,\ P_k] = 0\,.$$

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There is no mention in either of these cases of the Weyl-Heisenberg group for which the Hermitian representation of its algebra define the Heisenberg commutation relations fundamental to quantum physics.

Symplectic Symmetry on Phase Space

Consider phase space $\mathbb{P} \simeq \mathbb{R}^{2\,n}$ with an invariant symplectic metric $\omega = \zeta_{\alpha,\beta}\, d\, z^{\alpha}\, d\, z^{\beta}$

$$\varphi : \mathbb{P} \to \mathbb{P}, \quad \varphi^{*}(\omega) = \omega, \qquad \left[\frac{\partial \varphi^{\beta}}{\partial z^{\alpha}}\right] \in \mathcal{S}p(2\,n)$$

The symmetry including translations is $\mathcal{I}\mathcal{S}p(2\,n) \simeq \mathcal{S}p(2\,n) \otimes_{s} \mathcal{A}(2\,n)$.

The projective representations of $\mathcal{I}\mathcal{S}p(2\,n)$ are the unitary representations of its central extension

$$\mathcal{I}\check{\mathcal{S}}p(2\,n) \simeq \overline{\mathcal{S}p}(2\,n) \otimes_{s} \mathcal{H}(n)$$

$\mathcal{H}(n)$ is the Weyl-Heisenberg group. The Hermitian representations of its algebra corresponding to the unitary representation of the group are the Heisenberg position-momentum commutation relations.

Weyl-Heisenberg group

The abelian group $\mathcal{A}(2\,m)$ has algebra $[A_\alpha,\ A_\beta] = 0,\ \ \alpha,\ \beta,\ \ .. = 1,\ 2\,m$

Admits $m(2\,m-1)$ dimensional algebraic extension $I_{\alpha,\beta} = -I_{\beta,\alpha}$

$$[A_\alpha,\ A_\beta] = I_{\alpha,\beta},\quad [A_\alpha,\ I_{\alpha,\beta}] = 0,\quad [I_{\alpha,\beta},\ I_{\gamma,\kappa}] = 0$$

The Weyl-Heisenberg is the simply connected group has an algebra that is 1 dimensional extension of the abelian algebra

$$[A_\alpha,\ A_\beta] = \zeta_{\alpha,\beta}\, I,\quad \zeta_{\alpha,\beta} = -\zeta_{\beta,\alpha}$$

It is the semidirect product

$$\mathcal{H}(m)\ \simeq\ \mathcal{A}(m) \otimes_s \mathcal{A}(m+1)$$

It is a real matrix Lie group, $\Upsilon(w,\ \iota) \in \mathcal{H}(m)$ realized by $2\,n+2$ dimensional matrices

$$\Upsilon(z,\ \iota) = \begin{pmatrix} 1_{2\,n} & 0 & z \\ -z^{\mathrm{t}}\,\zeta & 1 & 2\,\iota \\ 0 & 0 & 1 \end{pmatrix},\ \ z \in \mathbb{R}^{2\,m},\ \iota \in \mathbb{R}$$

$$\Upsilon(z',\ \iota')\,\Upsilon(z,\ \iota) = \Upsilon\left(z' + z,\ \iota' + \iota + \frac{1}{2}\, z'^{\mathrm{t}}\,\zeta\, z\right),\ \ \Upsilon^{-1}(z,\ \iota) = \Upsilon(-z,\ -\iota)$$

Unitary representations of the Heisenberg group

The unitary representions of the Weyl-Heisenberg group $\mathcal{H}(n)$.

$$\{A_\alpha\} = \{P_i,\, Q_i\}, \qquad [P_i,\, Q_j] = \hbar\, \delta_{i,j}\, I,$$

From Mackey Theorems, Hilbert space is $L^2(\mathbb{R}^n, \mathbb{C})$. The Hermitian representation of the algebra with $\hat{Q}_i$ diagonal is

$$\hat{I}\, \psi(q) = \upsilon\, \psi(q), \;\; \hat{Q}_i\, \psi(q) = q_i\, \psi(q), \;\; \hat{P}_i\, \psi(q) = i\, \upsilon\, \hbar\, \frac{\partial}{\partial q_i}\, \psi(q), \quad \upsilon \in \mathbb{R}\backslash\{0\}$$

with commutators

$$\left[\hat{P}_i,\, \hat{Q}_j\right] = i\, \hbar\, \delta_{i,j}\, \hat{I},$$

Group transformation is

$$\tilde{\psi}(q) = \varrho(\Upsilon(\tilde{p},\, \tilde{q},\, \iota)\, \psi)\, (q) = e^{i\, \upsilon\left(\iota - \frac{1}{2}\, \tilde{q}\cdot\tilde{p}\right) + \tilde{p}\cdot q,}\, \psi(q - \tilde{q})$$

Weyl-Heisenberg automorphisms

Why does the $\mathcal{ISp}(2\,n)$ group constrain the abelian group central extension to the Weyl-Heisenberg group?

States transform as $|\tilde{\psi}\rangle = \varrho(g)\,|\psi\rangle$, ϱ a unitary representation of $g \in \mathcal{G}$. Generators transform as

$$\hat{Q}'_i = \varrho(g)\,\hat{Q}_i\,\varrho(g)^{-1},\;\; \hat{P}'_i = \varrho(g)\,\hat{P}_i\,\varrho(g)^{-1},\;\; \hat{I}' = \varrho(g)\,\hat{I}\,\varrho(g)^{-1} = \hat{I}$$

In quantum mechanics, we want the Weyl-Heisenberg commutation relations to hold at any point in the Hilbert space

$$i\,\hbar\,\delta_{i,j}\,\hat{I} = \left[\hat{P}_i,\,\hat{Q}_j\right] \Rightarrow \quad i\,\hbar\,\delta_{i,j}\,\hat{I}' = \left[\hat{P}'_i,\,\hat{Q}'_j\right]$$

Representation is faithful and therefore $\mathcal{G} \subset \mathcal{A}ut_{\mathcal{H}\,(n)}$

$$\mathcal{A}ut_{\mathcal{H}\,(n)} \simeq \mathcal{D} \otimes_s \overline{\mathcal{Sp}}(2\,n) \otimes_s \mathcal{H}(n) \simeq \mathcal{D}\check{\mathcal{S}}p(2\,n), \quad \mathcal{D} \simeq (\mathbb{R}\backslash\{0\},\,\times)$$

where

$$\mathcal{DSp}(2\,n) \simeq \mathcal{D} \otimes_s \mathcal{ISp}(2\,n)$$

Maximal quantum mechanical symmetry

Analysis applies also to extended phase space $\mathbb{P} = \mathbb{R}^{2n+2}$, $\{z^\alpha\} = \{p^i, q^i, \varepsilon, t\}$, $z \in \mathbb{P}$, with symplectic metric

$$\omega = \zeta_{\alpha,\beta}\, d z^\alpha \wedge d z^\beta = \delta_{i,j}\, d p^i \wedge d q^j + d t \wedge d \varepsilon$$

Symmetry group is $\mathcal{DSp}(2n+2)$ and

$$\check{\mathcal{D}}\mathcal{Sp}(2n+2) \simeq \mathcal{Aut}_{\mathcal{H}(n+1)}$$

Projective representations of $\mathcal{DSp}(2n+2)$ (that are the ordinary unitary representations of $\mathcal{Aut}_{\mathcal{H}(n+1)}$) is largest symmetry group with

a normal Weyl-Heisenberg subgroup with a representation of its Hermitian algebra that are the Heisenberg commutation relations.

$$\left[\hat{P}_i, \hat{Q}_j\right] = i\hbar\,\delta_{i,j}\,\hat{I}, \quad \left[\hat{T}, \hat{E}\right] = i\hbar\,\hat{I}$$

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This has the Weyl-Heisenberg group and Heisenberg commutation relations but there is no mention in this phase space discussion of relativity (line element giving an invariant definition of time.)

Relativity implications

Relativity reqires the homogeneous group leave invariant the invariant time line element

$$d\tau^2 = dt^2 - \frac{1}{c^2}dq^2 \underset{c\to\infty}{\rightarrow} dt^2$$

Consider the invariant Newtonian time, dt^2 .

Its invariance group is the affine group $\mathcal{IGL}(m-1)$. For extended phase $m = 2n+2$.

The group that leaves invariant both the Heisenberg commutation relations and the Newtonian time is

$$\mathcal{D} \otimes_s \mathcal{S}p(2n+2) \bigcap \mathcal{IGL}(2n+1) \simeq \mathcal{HS}p(2n) = \mathcal{S}p(2n) \otimes_s \mathcal{H}(n)$$

It is a matrix group with elements $\Gamma(\mathrm{A}, w, \iota) = \Gamma(1_{2n}, w, \iota)\,\Gamma(\mathrm{A}, 0, 0)$,

$$\Gamma(\mathrm{A}, 0, 0) = \begin{pmatrix} \mathrm{A} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathcal{S}p(2n),\ \mathrm{A} \in \mathcal{S}p(2n)$$

$$\Gamma(1_{2n}, w, r) = \begin{pmatrix} 1_{2n} & 0 & w \\ -w^{t}\zeta & 1 & r \\ 0 & 0 & 1 \end{pmatrix} \in \mathcal{H}(n),\ w \in \mathbb{R}^{2n},\ r \in \mathbb{R}$$

Diffeomorphisms with $\mathcal{HSp}(2\,n)$ symmetry

Let ϕ be diffeomorphism $\phi : \mathbb{P} \to \mathbb{P}$ leaving invariant the symplectic form ω and the degenerate line element $d\,t^2$

$$\phi^*(\omega) = \omega, \quad \phi^*\left(d\,t^2\right) = d\,t^2 \quad \Rightarrow \left[\frac{\partial\,\phi^\alpha}{\partial\,z^\beta}\right] = \Gamma(A,\, w,\, \iota) \in \mathcal{HSp}(2\,n)$$

As $\Gamma(\mathrm{A},\, w,\, \iota) = \Gamma(1_{2\,n},\, w,\, \iota)\,\Gamma(\mathrm{A},\, 0,\, 0)$, we can write $\phi = \varphi \circ \tilde{\varphi}$ where

$$\left[\frac{\partial\tilde{\varphi}^\alpha}{\partial\tilde{z}^\beta}\right] = \begin{pmatrix} \mathrm{A} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathcal{Sp}(2\,n)$$ are canonical transformations

$$\left[\frac{\partial\varphi^\alpha}{\partial z^\beta}\right] = \begin{pmatrix} 1_{2\,n} & 0 & w \\ -w^{\mathrm{t}}\,\zeta & 1 & 2\,r \\ 0 & 0 & 1 \end{pmatrix} \in \mathcal{H}(n)$$ is Hamilton's equations

Let $w = (f,\, v), \quad f,\, v \in \mathbb{R}^n$

$$\begin{pmatrix} \frac{\partial\varphi^i}{\partial p^j} & \frac{\partial\varphi^i}{\partial q^j} & \frac{\partial\varphi^i}{\partial\varepsilon} & \frac{\partial\varphi^i}{\partial t} \\ \frac{\partial\varphi^{n+i}}{\partial p^j} & \frac{\partial\varphi^{n+i}}{\partial q^j} & \frac{\partial\varphi^{n+i}}{\partial\varepsilon} & \frac{\partial\varphi^{n+i}}{\partial t} \\ \frac{\partial\varphi^{2n+1}}{\partial p^j} & \frac{\partial\varphi^{2n+1}}{\partial q^j} & \frac{\partial\varphi^{2n+1}}{\partial\varepsilon} & \frac{\partial\varphi^{2n+1}}{\partial t} \\ \frac{\partial\varphi^{2n+2}}{\partial p^j} & \frac{\partial\varphi^{2n+2}}{\partial q^j} & \frac{\partial\varphi^{2n+2}}{\partial\varepsilon} & \frac{\partial\varphi^{2n+1}}{\partial t} \end{pmatrix} = \begin{pmatrix} 1_n & 0 & 0 & f \\ 0 & 1_n & 0 & v \\ v & -f & 1 & r \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Hamilton’s equations

$$\begin{pmatrix} \frac{\partial\varphi^i}{\partial p^j} & \frac{\partial\varphi^i}{\partial q^j} & \frac{\partial\varphi^i}{\partial \varepsilon} & \frac{\partial\varphi^i}{\partial t} \\ \frac{\partial\varphi^{n+i}}{\partial p^j} & \frac{\partial\varphi^{n+i}}{\partial q^j} & \frac{\partial\varphi^{n+i}}{\partial \varepsilon} & \frac{\partial\varphi^{n+i}}{\partial t} \\ \frac{\partial\varphi^{2n+1}}{\partial p^j} & \frac{\partial\varphi^{2n+1}}{\partial q^j} & \frac{\partial\varphi^{2n+1}}{\partial \varepsilon} & \frac{\partial\varphi^{2n+1}}{\partial t} \\ \frac{\partial\varphi^{2n+2}}{\partial p^j} & \frac{\partial\varphi^{2n+2}}{\partial q^j} & \frac{\partial\varphi^{2n+2}}{\partial \varepsilon} & \frac{\partial\varphi^{2n+1}}{\partial t} \end{pmatrix} = \begin{pmatrix} 1_n & 0 & 0 & f \\ 0 & 1_n & 0 & v \\ v & -f & 1 & r \\ 0 & 0 & 0 & 1 \end{pmatrix} \Rightarrow$$

$$\varphi^{2n+2}(p, q, \varepsilon, t) = t$$

Hamilton's equations

$$\begin{pmatrix} \frac{\partial \varphi^i}{\partial p^j} & \frac{\partial \varphi^i}{\partial q^j} & \frac{\partial \varphi^i}{\partial \varepsilon} & \frac{\partial \varphi^i}{\partial t} \\ \frac{\partial \varphi^{n+i}}{\partial p^j} & \frac{\partial \varphi^{n+i}}{\partial q^j} & \frac{\partial \varphi^{n+i}}{\partial \varepsilon} & \frac{\partial \varphi^{n+i}}{\partial t} \\ \frac{\partial \varphi^{2n+1}}{\partial p^j} & \frac{\partial \varphi^{2n+1}}{\partial q^j} & \frac{\partial \varphi^{2n+1}}{\partial \varepsilon} & \frac{\partial \varphi^{2n+1}}{\partial t} \\ 0 & 0 & 0 & \frac{\partial \varphi^{2n+1}}{\partial t} \end{pmatrix} = \begin{pmatrix} 1_n & 0 & 0 & f \\ 0 & 1_n & 0 & v \\ v & -f & 1 & r \\ 0 & 0 & 0 & 1 \end{pmatrix} \Rightarrow \begin{matrix} \varphi^{2n+1}(p, q, \varepsilon, t) = \varepsilon + H(q, p, t) \\ \varphi^{2n+2}(p, q, \varepsilon, t) = t \end{matrix}$$

Hamilton’s equations

$$\begin{pmatrix} \frac{\partial\varphi^i}{\partial p^j} & \frac{\partial\varphi^i}{\partial q^j} & \frac{\partial\varphi^i}{\partial \varepsilon} & \frac{\partial\varphi^i}{\partial t} \\ \frac{\partial\varphi^{n+i}}{\partial p^j} & \frac{\partial\varphi^{n+i}}{\partial q^j} & \frac{\partial\varphi^{n+i}}{\partial \varepsilon} & \frac{\partial\varphi^{n+i}}{\partial t} \\ \frac{\partial\varphi^{2n+1}}{\partial p^j} & \frac{\partial\varphi^{2n+1}}{\partial q^j} & \frac{\partial\varphi^{2n+1}}{\partial \varepsilon} & \frac{\partial\varphi^{2n+1}}{\partial t} \\ 0 & 0 & 0 & \frac{\partial\varphi^{2n+1}}{\partial t} \end{pmatrix} = \begin{pmatrix} 1_n & 0 & 0 & f \\ 0 & 1_n & 0 & v \\ v & -f & 1 & r \\ 0 & 0 & 0 & 1 \end{pmatrix} \Rightarrow \begin{array}{l} \varphi^{n+i}(p, q, \varepsilon, t) = q^i + \xi^i(t) \\ \varphi^{2n+1}(p, q, \varepsilon, t) = \varepsilon + H(q, p, t) \\ \varphi^{2n+2}(p, q, \varepsilon, t) = t \end{array}$$

Hamilton's equations

$$\begin{pmatrix} \frac{\partial \varphi^i}{\partial p^j} & \frac{\partial \varphi^i}{\partial q^j} & \frac{\partial \varphi^i}{\partial \varepsilon} & \frac{\partial \varphi^i}{\partial t} \\ 0 & \frac{\partial \varphi^{n+i}}{\partial q^j} & 0 & \frac{\partial \varphi^{n+i}}{\partial t} \\ \frac{\partial \varphi^{2n+1}}{\partial p^j} & \frac{\partial \varphi^{2n+1}}{\partial q^j} & \frac{\partial \varphi^{2n+1}}{\partial \varepsilon} & \frac{\partial \varphi^{2n+1}}{\partial t} \\ 0 & 0 & 0 & \frac{\partial \varphi^{2n+1}}{\partial t} \end{pmatrix} = \begin{pmatrix} 1_n & 0 & 0 & f \\ 0 & 1_n & 0 & v \\ v & -f & 1 & r \\ 0 & 0 & 0 & 1 \end{pmatrix} \Rightarrow \begin{array}{l} \varphi^i(p, q, \varepsilon, t) = p^i + \pi^i(t) \\ \varphi^{n+i}(p, q, \varepsilon, t) = q^i + \xi^i(t) \\ \varphi^{2n+1}(p, q, \varepsilon, t) = \varepsilon + H(q, p, t) \\ \varphi^{2n+2}(p, q, \varepsilon, t) = t \end{array}$$

Hamilton’s equations

$$\begin{pmatrix} 1_n & 0 & 0 & \frac{\partial \pi^i(t)}{\partial t} \\ 0 & 1_n & 0 & \frac{\partial \xi^i(t)}{\partial t} \\ \frac{\partial H(p,q,t)}{\partial p^j} & \frac{\partial H(p,q,t)}{\partial q^j} & 1 & \frac{\partial H(p,q,t)}{\partial t} \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1_n & 0 & 0 & f \\ 0 & 1_n & 0 & v \\ v & -f & 1 & r \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\varphi^i(p, q, \varepsilon, t) = p^i + \pi^i(t)$$
$$\varphi^{n+i}(p, q, \varepsilon, t) = q^i + \xi^i(t)$$
$$\varphi^{2n+1}(p, q, \varepsilon, t) = \varepsilon + H(q, p, t)$$
$$\varphi^{2n+2}(p, q, \varepsilon, t) = t$$

$$\frac{d\pi^i(t)}{dt} = f(p, q, t) = -\frac{\partial H(p, q, t)}{\partial q^j},$$

$$\frac{d\xi^i(t)}{dt} = v(p, q, t) = \frac{\partial H(p, q, t)}{\partial q^j}, \quad \frac{\partial H(p, q, t)}{\partial q^j} = r(p, q, t)$$

Hamilton's equations

$$\begin{pmatrix} 1_n & 0 & 0 & \frac{\partial \pi^i(t)}{\partial t} \\ 0 & 1_n & 0 & \frac{\partial \xi^i(t)}{\partial t} \\ \frac{\partial H(p,q,t)}{\partial p^j} & \frac{\partial H(p,q,t)}{\partial q^j} & 1 & \frac{\partial H(p,q,t)}{\partial t} \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1_n & 0 & 0 & f \\ 0 & 1_n & 0 & v \\ v & -f & 1 & r \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\varphi^i (p, q, \varepsilon, t) = p^i + \pi^i (t)$$

$$\varphi^{n+i} (p, q, \varepsilon, t) = q^i + \xi^i (t)$$

$$\varphi^{2n+1}(p, q, \varepsilon, t) = \varepsilon + H(q, p, t)$$

$$\varphi^{2n+2} (p, q, \varepsilon, t) = t$$

$$\frac{d\pi^i(t)}{dt} = f(p, q, t) = -\frac{\partial H(p, q, t)}{\partial q^j},$$

$$\frac{d\xi^i(t)}{dt} = v(p, q, t) = \frac{\partial H(p, q, t)}{\partial q^j}, \quad \frac{\partial H(p, q, t)}{\partial q^j} = r(p, q, t)$$

$\mathcal{HS}p(2n) \subset \mathcal{S}p(2n+2)$ is the general local noninertial symmetry of Hamilton's equations leaving time invariant.

$\mathcal{S}p(2n)$ is the usual symplectic symmetry defining canonical transformations

$\mathcal{H}(n)$ is parameterized by velocity, force and power (power is the central element).

Noncommutativity simply means that two transformations that are noninertial do not commute

Inertial subgroup $\mathcal{E}(n)$ that is parameterized by rotations and velocity is a subgroup of $\mathcal{HS}p(n)$.

Quantum Mechanical Symmetry

The projective representation of

$$\mathcal{IHS}p(2n) \simeq \mathcal{HS}p(2n) \otimes_s \mathcal{A}(2n+2)$$

are the ordinary unitary representations of

$$\begin{aligned} \mathcal{I}\check{\mathcal{H}}\mathcal{S}p\,(2n) &\simeq \overline{\mathcal{HS}p}\,(2n) \otimes_s \mathcal{H}\,(n+1) \\ &\simeq \overline{\mathcal{S}p}(2n) \otimes_s \mathcal{H}(n) \otimes_s \mathcal{H}(n+1) \end{aligned}$$

If we require invariance of length, restricts $\mathcal{S}p(2n)$ to $\mathcal{SO}(n)$

$$\mathcal{IH}a(n) \simeq \mathcal{H}a(n) \otimes_s \mathcal{A}(2n+2), \quad \mathcal{H}a(n) \simeq \mathcal{SO}(n) \otimes_s \mathcal{H}(n)$$

With the central extension

$$\begin{aligned} \mathcal{I}\check{\mathcal{H}}a(n) &\simeq \overline{\mathcal{H}a}\,(2n) \otimes_s (\mathcal{H}(n+1) \otimes \mathcal{A}(2)) \\ &\simeq \overline{\mathcal{SO}}(2n) \otimes_s \mathcal{H}(n) \otimes_s (\mathcal{H}(n+1) \otimes \mathcal{A}(2)) \end{aligned}$$

The Galilei group is an inertial subgroup.